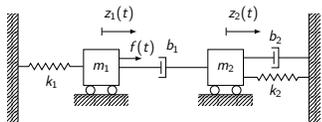


Esercizio 1



Rappresentazione interna o di stato con $u(t) = f(t)$ e $y(t) = [z_1 \ z_2]^T$?

$f(t) = \text{forza esterna} = \text{input}$

$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \text{uscita}$

Rappresentazione di sistema:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \quad \begin{array}{l} x_1(t) = z_1(t) \\ x_2(t) = z_2(t) \\ x_3(t) = \dot{z}_1(t) \\ x_4(t) = \dot{z}_2(t) \end{array}$$

Legge di Newton:

$$1) \ m_1 \ddot{z}_1 = f - k_1 z_1 - b_1 (\dot{z}_1 - \dot{z}_2)$$

$$2) \ m_2 \ddot{z}_2 = -k_2 z_2 - b_2 \dot{z}_2 + b_1 (\dot{z}_1 - \dot{z}_2)$$

$$\dot{x}_1 = \dot{z}_1 = x_3$$

$$x_2 = z_2 = x_4$$

$$\dot{x}_3 = \ddot{z}_1 = \frac{1}{m_1} (f - k_1 z_1 - b_1 \dot{z}_1 + b_1 \dot{z}_2) = \frac{1}{m_1} u - \frac{k_1}{m_1} x_1 - \frac{b_1}{m_1} x_3 + \frac{b_1}{m_1} x_4$$

$$\dot{x}_4 = \ddot{z}_2 = \frac{1}{m_2} (-k_2 z_2 - b_2 \dot{z}_2 + b_1 \dot{z}_1 - b_1 \dot{z}_2) = -\frac{k_2}{m_2} x_2 - \frac{b_1 + b_2}{m_2} x_4 + \frac{b_1}{m_2} x_3$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & 0 & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ 0 & -\frac{k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1 + b_2}{m_2} \end{bmatrix}}_F x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix}}_G u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_H x + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_J u$$

$$\dot{x}(t) = Fx(t) + Gu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha & \alpha-2 & 0 \\ 2 & 0 & 2-\alpha^2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \quad \alpha \in \mathbb{R}$$

$$y(t) = Hx(t) = [1 \quad -2 \quad 0] x(t)$$

1. Forma di Jordan F_J e i modi del sistema al variare di $\alpha \in \mathbb{R}$?

2. Funzione di trasferimento $W(s)$ al variare di $\alpha \in \mathbb{R}$?

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha & \alpha-2 & 0 \\ 2 & 0 & 2-\alpha^2 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad H = [1 \quad -2 \quad 0]$$

$$= \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$$

1) Forma di Jordan F_J e modi del sistema:

$$\lambda(F) = \lambda(F_{11}) \cup \lambda(F_{22})$$

(i) Calcolo autovalori F :

$$\Delta_{F_{11}}(\lambda) = \det(\lambda I - F_{11}) = \det \begin{bmatrix} \lambda & -1 \\ -2\alpha & \lambda - \alpha + 2 \end{bmatrix} = \lambda(\lambda - \alpha + 2) - 2\alpha$$

$$= \lambda^2 + (2 - \alpha)\lambda - 2\alpha$$

$$\lambda(F) = \{ \alpha, -2, 2 - \alpha^2 \}$$

1 $\alpha = -2$: $\lambda_1 = -2, v_1 = 3$

2 $-2 = 2 - \alpha^2 \Rightarrow \alpha^2 = 4 \Rightarrow \alpha = \pm 2$

$\alpha = 2$: $\lambda_1 = -2, v_1 = 1 = g_1$
 $\lambda_2 = 2, v_1 = 2$

3 $\alpha = 2 - \alpha^2 \Rightarrow \alpha^2 + \alpha - 2 = 0$

$$(\alpha - 1)(\alpha + 2) = 0$$

$\alpha = 1$: $\lambda_1 = -2, v_1 = 1 = g_1$
 $\lambda_2 = 1, v_1 = 2$

$$\lambda_{1,2} = \frac{-2 + \alpha \pm \sqrt{(2 - \alpha)^2 + 8\alpha}}{2}$$

$$= \frac{-2 + \alpha \pm \sqrt{(\alpha + 2)^2}}{2}$$

$$= \frac{-2 + \alpha \pm (\alpha + 2)}{2}$$

$$\frac{-\cancel{\alpha} + \alpha + \alpha + \cancel{\alpha}}{2} = \alpha$$

$$\frac{-2 + \alpha - \alpha - 2}{2} = -2$$

4 $\alpha \neq \{ \pm 2, 1 \}$: $\left. \begin{array}{l} \lambda_1 = \alpha \quad v_1 = 1 = g_1 \\ \lambda_2 = -2 \quad v_2 = 1 = g_2 \\ \lambda_3 = 2 - \alpha^2 \quad v_3 = 1 = g_3 \end{array} \right\} F \text{ diagonalizzabile}$

(ii) Calcolo delle molteplicità geometriche:

1 $\alpha = -2$: $\lambda_1 = -2$ $v_1 = 3$

$$g_1 = 3 - \text{rank}(\lambda_1 I - F) = 3 - \text{rank} \begin{bmatrix} -2 & -1 & 0 \\ -2\alpha & -\alpha & 0 \\ -2 & 0 & -4 + \alpha^2 \end{bmatrix}$$
$$= 3 - \text{rank} \begin{bmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} = 3 - 2 = 1$$

2 $\alpha = 2$: $\lambda_1 = -2$ $v_1 = g_1 = 1$
 $\lambda_2 = 2$ $v_2 = 2$

$$g_2 = 3 - \text{rank}(\lambda_2 I - F) = 3 - \text{rank} \begin{bmatrix} 2 & -1 & 0 \\ -2\alpha & 4 - \alpha & 0 \\ -2 & 0 & \alpha^2 \end{bmatrix}$$
$$= 3 - \text{rank} \begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} = 3 - 2 = 1$$

\downarrow \downarrow \downarrow
 v_1 v_2 v_3

$$v_1 = -2v_2 - \frac{1}{2}v_3$$

3 $\alpha = 1$: $\lambda_1 = -2$ $v_1 = g_1 = 1$
 $\lambda_2 = 1$ $v_2 = 2$

$$g_2 = 3 - \text{rank}(\lambda_2 I - F) = 3 - \text{rank} \begin{bmatrix} 1 & -1 & 0 \\ -2\alpha & 3 - \alpha & 0 \\ -2 & 0 & -1 + \alpha^2 \end{bmatrix}$$
$$= 3 - \text{rank} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} = 3 - 2 = 1$$

(iii) Calcolo F_J e modi:

1 $\alpha = -2$: $\lambda_1 = -2$, $v_1 = 3$, $g_1 = 1$

$$F_J = \begin{bmatrix} -2 & 1 \\ & -2 & 1 \\ & & -2 \end{bmatrix} \quad \text{modi: } e^{-2t}, te^{-2t}, \frac{t^2}{2}e^{-2t}$$

2 $\alpha = 2$: $\lambda_1 = -2$, $v_1 = g_1 = 1$
 $\lambda_2 = 2$, $v_2 = 2$, $g_2 = 1$

$$F_J = \begin{bmatrix} -2 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix} \quad \text{modi: } e^{-2t}, e^{2t}, te^{2t}$$

3 $\alpha = 1$: $\lambda_1 = -2$, $v_1 = g_1 = 1$
 $\lambda_2 = 1$, $v_2 = 2$, $g_2 = 1$

$$F_J = \begin{bmatrix} -2 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \quad \text{modi: } e^{-2t}, e^t, te^t$$

4 $\alpha \neq \{-2, 1\}$ $\lambda_1 = -2$ $v_1 = g_1 = 1$
 $\lambda_2 = \alpha$ $v_2 = g_2 = 1$
 $\lambda_3 = 2 - \alpha^2$ $v_3 = g_3 = 1$

$$F_J = \begin{bmatrix} -2 & & \\ & \alpha & \\ & & 2 - \alpha^2 \end{bmatrix} \quad \text{modi: } e^{-2t}, e^{\alpha t}, e^{(2 - \alpha^2)t}$$

2) Calcolo di $W(s)$

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha & \alpha-2 & 0 \\ 2 & 0 & 2-\alpha^2 \end{bmatrix} \\ = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$$

$$G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}$$

$$W(s) = H (sI - F)^{-1} G$$

$$= \begin{bmatrix} H_1 & 0 \end{bmatrix} \begin{bmatrix} sI - F_{11} & 0 \\ -F_{21} & sI - F_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} H_1 & 0 \end{bmatrix} \begin{bmatrix} (sI - F_{11})^{-1} & 0 \\ * & (sI - F_{22})^{-1} \end{bmatrix} \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} H_1 (sI - F_{11})^{-1} & 0 \end{bmatrix} \begin{bmatrix} G_1 \\ 0 \end{bmatrix} = H_1 (sI - F_{11})^{-1} G_1$$

$$= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s & -1 \\ -2\alpha & s-\alpha+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \end{bmatrix} \frac{1}{s(s-\alpha+2)-2\alpha} \begin{bmatrix} s-\alpha+2 & 1 \\ 2\alpha & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{-s + 3 - 5\alpha}{s^2 + (2-\alpha)s - 2\alpha}$$

N.B.

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ * & C^{-1} \end{bmatrix}$$

Esercizio 3

$$\dot{x}(t) = Fx(t) = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} x(t)$$

$$y(t) = Hx(t) = [1 \ 1 \ 1] x(t)$$

1. Per $\alpha = \beta = 1$, $x(0) \neq 0$ tali che $y(t)$ non è divergente?

2. Per $\alpha = 1$ e $\beta = -1$, $x(0) \neq 0$ tali che $y(t)$ è limitata?

G. Baggio

Lez. 8: Esercizi di riepilogazione Parte I

12 Marzo 2021

$$F = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}$$

$$H = [1 \ 1 \ 1]$$

1) $\alpha = \beta = 1$

$$y(t) = H e^{Ft} x(0)$$

Calcolo di e^{Ft} : $e^{Ft} = \begin{bmatrix} e^{F_{11}t} & 0 \\ 0 & e^{F_{22}t} \end{bmatrix} \rightarrow e^{F_{11}t}$

Calcolo $e^{F_{11}t} = T e^{F_0 t} T^{-1}$ $T^{-1} F T = F_D$

(i) Autovalori di F_{11} : $\Delta_{F_{11}}(\lambda) = \det(\lambda I - F_{11}) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1$

$$\left. \begin{array}{l} \lambda_1 = 1 \quad v_1 = 1 = g_1 \\ \lambda_2 = -1 \quad v_2 = 1 = g_2 \end{array} \right\} F_{11} \text{ diagonalizzabile}$$

(ii) Autovettori di F_{11} :

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ autovettore di } \lambda_1 = 1: (\lambda_1 I - F_{11})v = 0 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} v_1 = v_2 \\ \text{"} \end{cases} \quad v = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ autovettore di } \lambda_2 = -1: (\lambda_2 I - F_{11})v = 0 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} v_1 = -v_2 \\ \text{"} \end{cases} \quad v = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$

Matrice di cambio base: $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$e^{Ft} = \begin{bmatrix} e^{F_{11}t} & 0 \\ 0 & e^{F_{22}t} \end{bmatrix} = \begin{bmatrix} T e^{F_{11,0}t} T^{-1} & 0 \\ 0 & e^{2t} \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$T e^{F_{11,0}t} T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$$

$$y(t) = H e^{Ft} x(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ e^t - e^{-t} & e^t + e^{-t} & 0 \\ 0 & 0 & 2e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^t & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$= e^t x_{0,1} + e^t x_{0,2} + e^{2t} x_{0,3} \quad x(0) = \begin{bmatrix} \gamma \\ -\gamma \\ 0 \end{bmatrix} \quad \begin{array}{l} \gamma \in \mathbb{R} \\ \gamma \neq 0 \end{array}$$

$$= 0 \quad \leftarrow$$

$$2) \alpha = 1, \beta = -1$$

$$F = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^{Ft} = \begin{bmatrix} e^{F_{11}t} & 0 \\ 0 & e^{F_{22}t} \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$y(t) = He^{Ft}x(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$= \begin{bmatrix} \cos t - \sin t & \cos t + \sin t & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$x(0) = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{bmatrix} \quad \gamma_1, \gamma_2 \in \mathbb{R}$$

(almeno uno tra γ_1 e $\gamma_2 \neq 0$)

$$x(t+1) = Fx(t) + Gu(t) = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = x(t)$$

1. Modi del sistema e loro carattere?

2. Matrice di trasferimento $W(z)$?

3. Evoluzione del sistema per $x(0) = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix}$ e $u(t) = \delta(t)$, $t \geq 0$?

$$F = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1) Modi e carattere.

$$\text{Autovalori di } F: \lambda_1 = -3 \quad v_1 = 1 = g_1 \\ \lambda_2 = 2 \quad v_2 = 1 = g_2$$

$$\text{Forma di Jordan: } F_J = \begin{bmatrix} -3 & \\ & 2 \end{bmatrix} \quad \text{modi: } (-3)^t, 2^t \text{ divergenti}$$

$$2) W(z) = H(zI - F)^{-1}G$$

$$= \begin{bmatrix} z+3 & -1 \\ 0 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(z+3)(z-2)} \begin{bmatrix} z-2 & 1 \\ 0 & z+3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(z+3)(z-2)} \begin{bmatrix} 1 \\ z+3 \end{bmatrix} = \begin{bmatrix} \frac{1}{(z+3)(z-2)} \\ \frac{1}{z-2} \end{bmatrix}$$

$$3) x(0) = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} \quad u(t) = \delta(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$$

$$X(z) = \underbrace{z(zI - F)^{-1}x(0)}_{X_h(z)} + \underbrace{W(z)U(z)}_{X_f(z)} \quad \frac{1}{10} (-3)^t \uparrow z^{-1}$$

$$X_h(z) = \frac{z}{(z+3)(z-2)} \begin{bmatrix} z-2 & 1 \\ 0 & z+3 \end{bmatrix} \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \frac{z}{(z+3)(z-2)} \begin{bmatrix} \frac{1}{10}(z-2) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \frac{z}{z+3} \\ 0 \end{bmatrix}$$

$$X_f(z) = W(z) U(z) = \begin{bmatrix} \frac{1}{(z+3)(z-2)} \\ \frac{1}{z-2} \end{bmatrix} = \begin{bmatrix} X_{f,1}(z) \\ X_{f,2}(z) \end{bmatrix}$$

Calcolo di $x_{f,1}(t)$:

$$X'_{f,1}(z) = \frac{X_{f,1}(z)}{z} = \frac{1}{z(z+3)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z+3} + \frac{A_3}{z-2}$$

$$A_1 = -1/6, \quad A_2 = 1/15, \quad A_3 = 1/10$$

$$X_{f,1}(z) = z X'_{f,1}(z) = A_1 + \frac{A_2 z}{z+3} + \frac{A_3 z}{z-2}$$

$$\begin{array}{ccc} \mathcal{Z}^{-1} \downarrow & \mathcal{Z}^{-1} \downarrow & \mathcal{Z}^{-1} \downarrow \\ A_1 \delta(t) & A_2 (-3)^t & A_3 2^t \end{array}$$

Calcolo di $x_{f,2}(t)$:

$$X'_{f,2}(z) = \frac{X_{f,2}(z)}{z} = \frac{1}{z(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-2} \quad A_1 = -1/2, \quad A_2 = 1/2$$

$$X_{f,2} = z X'_{f,2}(z) = A_1 + \frac{A_2 z}{z-2}$$

$$\begin{array}{ccc} \mathcal{Z}^{-1} \downarrow & \mathcal{Z}^{-1} \downarrow \\ A_1 \delta(t) & A_2 2^t \end{array}$$