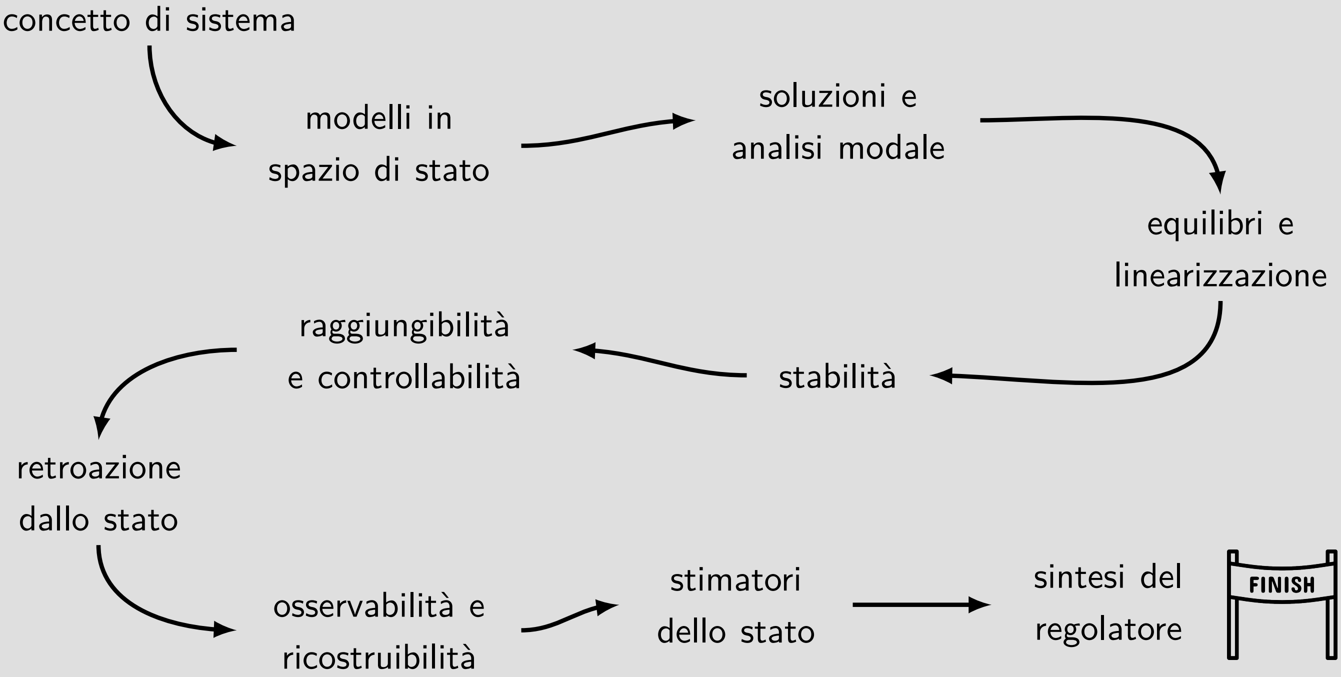
# Teoria dei Sistemi e Controllo Ottimo e Adattativo (C. I.)
# Teoria dei Sistemi (Mod. A)

Docente: Giacomo Baggio

Lez. 8: Esercizi di ricapitolazione Parte I

Corso di Laurea Magistrale in Ingegneria Meccatronica

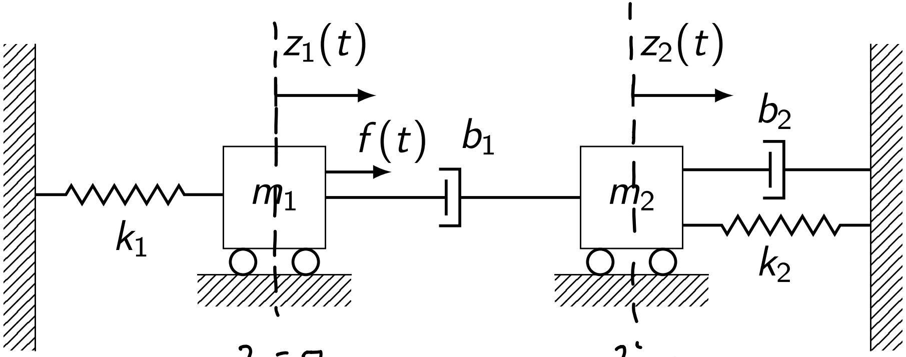
A.A. 2020-2021


concetto di sistema
modelli in spazio di stato
soluzioni e analisi modale
equilibri e linearizzazione
stabilità
raggiungibilità e controllabilità
retroazione dallo stato
osservabilità e ricostruibilità
stimatori dello stato
sintesi del regolatore
FINISH

# In questa lezione

- ▷ Esercizio 1: modelli in spazio di stato
- ▷ Esercizio 2: forma di Jordan e analisi modale a t.c.
- ▷ Esercizio 3: analisi modale ed evoluzione libera a t.c.
- ▷ Esercizio 4: analisi modale ed evoluzione forzata a t.d.

# Esercizio 1




$z_1 = 0$ $z_2 = 0$

molle sono a riposo

Rappresentazione interna o di stato con $u(t) = f(t)$ e $y(t) = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^\top$?

note

# Esercizio 1: soluzione

Variabili: $x(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix}$, $u(t) = f(t)$, $y(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$

Matrici: $F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & 0 & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ 0 & -\frac{k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $J = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

# Esercizio 2 [riadattato da Es. 1 tema d'esame 1 Febbraio 2012]

$$\dot{x}(t) = Fx(t) + Gu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha & \alpha - 2 & 0 \\ 2 & 0 & 2 - \alpha^2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \quad \alpha \in \mathbb{R}$$

$$y(t) = Hx(t) = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} x(t)$$

1. Forma di Jordan $F_J$ e i modi del sistema al variare di $\alpha \in \mathbb{R}$?

2. Funzione di trasferimento $W(s)$ al variare di $\alpha \in \mathbb{R}$?

note

# Esercizio 2: soluzione

1. $$F_J = \begin{cases} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} & \alpha = 1, \ \text{modi: } e^{t}, te^{t}, e^{-2t} \\ \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} & \alpha = 2, \ \text{modi: } e^{-2t}, te^{-2t}, e^{2t} \\ \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} & \alpha = -2, \ \text{modi: } e^{-2t}, te^{-2t}, \frac{t^2}{2}e^{-2t} \\ \begin{bmatrix} 2-\alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2 \end{bmatrix} & \alpha \neq 1, 2, -2, \ \text{modi: } e^{(2-\alpha^2)t}, e^{\alpha t}, e^{-2t} \end{cases}$$

2. $$W(s) = \frac{(3-5\alpha)-s}{(s-\alpha)(s+2)}$$

# Esercizio 3

$$\dot{x}(t) = Fx(t) = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} x(t)$$

$$y(t) = Hx(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(t)$$

1. Per $\alpha = \beta = 1$, $x(0) \neq 0$ tali che $y(t)$ non è divergente?
2. Per $\alpha = 1$ e $\beta = -1$, $x(0) \neq 0$ tali che $y(t)$ è limitata?

note

# Esercizio 3: soluzione

1. $x(0) = \begin{bmatrix} \gamma & -\gamma & 0 \end{bmatrix}^\top, \gamma \in \mathbb{R}$

2. $x(0) = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 \end{bmatrix}^\top, \gamma_1, \gamma_2 \in \mathbb{R}$

# Esercizio 4 [riadattato da Es. 3 tema d'esame 24 Giugno 2019]

$$x(t+1) = Fx(t) + Gu(t) = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = x(t)$$

1. Modi del sistema e loro carattere?

2. Matrice di trasferimento $W(z)$?

3. Evoluzione del sistema per $x(0) = \begin{bmatrix} \frac{1}{10} \\ 0 \end{bmatrix}$ e $u(t) = \delta(t)$, $t \geq 0$?

note

# Esercizio 4: soluzione

1. $(-3)^t, 2^t$, entrambi divergenti

2. $W(z) = \begin{bmatrix} \frac{1}{(z+3)(z-2)} \\ \frac{1}{z-2} \end{bmatrix}$

3. $y(t) = \begin{bmatrix} \frac{1}{10}2^t + \frac{1}{6}(-3)^t - \frac{1}{6}\delta(t) \\ \frac{1}{2}2^t - \frac{1}{2}\delta(t) \end{bmatrix}$, $t \geq 0$

# Teoria dei Sistemi e Controllo Ottimo e Adattativo (C. I.)
# Teoria dei Sistemi (Mod. A)

Docente: Giacomo Baggio

Lez. 8: Esercizi di ricapitolazione Parte I

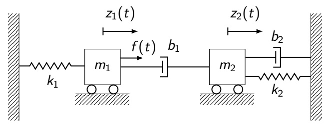
Corso di Laurea Magistrale in Ingegneria Meccatronica

A.A. 2020-2021

✉ baggio@dei.unipd.it

baggiogi.github.io

## Esercizio 1




Rappresentazione interna o di stato con $u(t) = f(t)$ e $y(t) = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^\top$?



$f(t) =$ forza esterna $=$ input

$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} =$ uscita

Rappresentazione di sistema:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \qquad \begin{aligned} x_1(t) &= z_1(t) \\ x_2(t) &= z_2(t) \\ x_3(t) &= \dot z_1(t) \\ x_4(t) &= \dot z_2(t) \end{aligned}$$

Legge di Newton:

1) $m_1 \ddot z_1 = f - k_1 z_1 - b_1(\dot z_1 - \dot z_2)$

2) $m_2 \ddot z_2 = -k_2 z_2 - b_2 \dot z_2 + b_1(\dot z_1 - \dot z_2)$

---

$\dot x_1 = \dot z_1 = x_3$

$\dot x_2 = \dot z_2 = x_4$

$$\dot x_3 = \ddot z_1 = \frac{1}{m_1}\left(f - k_1 z_1 - b_1 \dot z_1 + b_1 \dot z_2\right) = \frac{1}{m_1} u - \frac{k_1}{m_1} x_1 - \frac{b_1}{m_1} x_3 + \frac{b_1}{m_1} x_4$$

$$\dot x_4 = \ddot z_2 = \frac{1}{m_2}\left(-k_2 z_2 - b_2 \dot z_2 + b_1 \dot z_1 - b_1 \dot z_2\right) = -\frac{k_2}{m_2} x_2 - \frac{b_1+b_2}{m_2} x_4 + \frac{b_1}{m_2} x_3$$

$$\begin{cases} \dot x = \overbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & 0 & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ 0 & -\frac{k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}}^{F} x + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix}}^{G} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{H} x + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{J} u \end{cases}$$

**Esercizio 2** [riadattato da Es. 1 tema d'esame 1 Febbraio 2012]

$$\dot{x}(t) = Fx(t) + Gu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha & \alpha-2 & 0 \\ 2 & 0 & 2-\alpha^2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \quad \alpha \in \mathbb{R}$$

$$y(t) = Hx(t) = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} x(t)$$

1. Forma di Jordan $F_J$ e i modi del sistema al variare di $\alpha \in \mathbb{R}$?
2. Funzione di trasferimento $W(s)$ al variare di $\alpha \in \mathbb{R}$?



$$F = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 2\alpha & \alpha-2 & 0 \\ \hline 2 & 0 & 2-\alpha^2 \end{array}\right] = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}$$

1) Forma di Jordan $F_J$ e modi del sistema:

$$\lambda(F) = \lambda(F_{11}) \cup \lambda(F_{22})$$

(i) Calcolo autovalori $F$:

$$\underline{\Delta_{F_{11}}(\lambda) = \det(\lambda I - F_{11})} = \det \begin{bmatrix} \lambda & -1 \\ -2\alpha & \lambda-\alpha+2 \end{bmatrix} = \lambda(\lambda-\alpha+2) - 2\alpha = \lambda^2 + (2-\alpha)\lambda - 2\alpha$$

$$\lambda_{1,2} = \frac{-2+\alpha \pm \sqrt{(2-\alpha)^2 + 8\alpha}}{2} = \frac{-2+\alpha \pm \sqrt{(\alpha+2)^2}}{2} = \frac{-2+\alpha \pm (\alpha+2)}{2}$$

$$\frac{\cancel{-2}+\alpha+\alpha\cancel{+2}}{2} = \alpha \qquad \frac{-2\cancel{+\alpha}\cancel{-\alpha}-2}{2} = -2$$

$$\lambda(F) = \{\alpha, -2, 2-\alpha^2\}$$

$\boxed{1}$ $\underline{\alpha = -2}$ : $\lambda_1 = -2$, $\nu_1 = 3$

$\boxed{2}$ $-2 = 2-\alpha^2 \Rightarrow \alpha^2 = 4 \Rightarrow \alpha = \pm 2$

$\underline{\alpha = 2}$ : $\lambda_1 = -2$, $\nu_1 = 1 = g_1$
$\lambda_2 = 2$, $\nu_1 = 2$

$\boxed{3}$ $\alpha = 2-\alpha^2 \Rightarrow \alpha^2 + \alpha - 2 = 0$
$(\alpha-1)(\alpha+2) = 0$

$\underline{\alpha = 1}$ : $\lambda_1 = -2$, $\nu_1 = 1 = g_1$
$\lambda_2 = 1$, $\nu_1 = 2$

$\boxed{4}$ $\alpha \notin \{\pm 2, 1\}$ :

$$\left.\begin{array}{ll} \lambda_1 = \alpha & \nu_1 = 1 = g_1 \\ \lambda_2 = -2 & \nu_2 = 1 = g_2 \\ \lambda_3 = 2-\alpha^2 & \nu_3 = 1 = g_3 \end{array}\right\} F \text{ diagonalizzabile}$$

(ii) Calcolo delle molteplicità geometriche:

$\boxed{1}$ $\underline{\alpha=-2}$: $\lambda_1=-2$ $\quad \nu_1=3$

$$g_1 = 3-\text{rank}(\lambda_1 I - F) = 3-\text{rank}\begin{bmatrix} -2 & -1 & 0 \\ -2\alpha & -\alpha & 0 \\ -2 & 0 & -4+\alpha^2 \end{bmatrix} = 3-\text{rank}\begin{bmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} = 3-2=1$$

$\boxed{2}$ $\underline{\alpha=2}$: $\lambda_1=-2$ $\quad \nu_1=g_1=1$

$\lambda_2=2$ $\quad \nu_2=2$

$$g_2 = 3-\text{rank}(\lambda_2 I - F) = 3-\text{rank}\begin{bmatrix} 2 & -1 & 0 \\ -2\alpha & 4-\alpha & 0 \\ -2 & 0 & \alpha^2 \end{bmatrix} = 3-\text{rank}\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} = 3-2=1$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array}$$

$$v_1 = -2v_2 - \frac{1}{2}v_3$$

$\boxed{3}$ $\underline{\alpha=1}$: $\lambda_1=-2$ $\quad \nu_1=g_1=1$

$\lambda_2=1$ $\quad \nu_2=2$

$$g_2 = 3-\text{rank}(\lambda_2 I - F) = 3-\text{rank}\begin{bmatrix} 1 & -1 & 0 \\ -2\alpha & 3-\alpha & 0 \\ -2 & 0 & -1+\alpha^2 \end{bmatrix} = 3-\text{rank}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} = 3-2=1$$

(iii) Calcolo $F_J$ e modi:

$\boxed{1}$ $\underline{\alpha=-2}$: $\lambda_1=-2$, $\nu_1=3$, $g_1=1$

$$F_J=\begin{bmatrix} -2 & 1 & \\ & -2 & 1 \\ & & -2 \end{bmatrix} \qquad \text{modi: } e^{-2t},\ te^{-2t},\ \frac{t^2}{2}e^{-2t}$$

$\boxed{2}$ $\underline{\alpha=2}$: $\lambda_1=-2$, $\nu_1=g_1=1$
$\lambda_2=2$, $\nu_2=2$, $g_2=1$

$$F_J=\left[\begin{array}{c:cc} -2 & & \\ \hdashline & 2 & 1 \\ & & 2 \end{array}\right] \qquad \text{modi: } e^{-2t},\ e^{2t},\ te^{2t}$$

$\boxed{3}$ $\underline{\alpha=1}$: $\lambda_1=-2$, $\nu_1=g_1=1$
$\lambda_2=1$, $\nu_2=2$, $g_2=1$

$$F_J=\left[\begin{array}{c:cc} -2 & & \\ \hdashline & 1 & 1 \\ & & 1 \end{array}\right] \qquad \text{modi: } e^{-2t},\ e^{t},\ te^{t}$$

$\boxed{4}$ $\alpha\neq\{\pm2,1\}$ $\quad \lambda_1=-2 \quad \nu_1=g_1=1$
$\lambda_2=\alpha \quad \nu_2=g_2=1$
$\lambda_3=2-\alpha^2 \quad \nu_3=g_3=1$

$$F_J=\begin{bmatrix} -2 & & \\ & \alpha & \\ & & 2-\alpha^2 \end{bmatrix} \qquad \text{modi: } e^{-2t},\ e^{\alpha t},\ e^{(2-\alpha^2)t}$$

2) Calcolo di $W(s)$

$$F = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 2\alpha & \alpha-2 & 0 \\ \hline 2 & 0 & 2-\alpha^2 \end{array}\right] = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \qquad G = \begin{bmatrix} \overbrace{\begin{matrix} 1 \\ 1 \end{matrix}}^{G_1} \\ \hline 0 \end{bmatrix} \qquad H = [\underbrace{1 \ -2}_{H_1} \mid 0]$$

$$\begin{aligned} W(s) &= H(sI-F)^{-1}G \\ &= [H_1 \ \ 0] \begin{bmatrix} sI-F_{11} & 0 \\ -F_{21} & sI-F_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \\ &= [H_1 \ \ 0] \begin{bmatrix} (sI-F_{11})^{-1} & 0 \\ * & (sI-F_{22})^{-1} \end{bmatrix} \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \\ &= [H_1(sI-F_{11})^{-1} \ \ 0] \begin{bmatrix} G_1 \\ 0 \end{bmatrix} = H_1(sI-F_{11})^{-1}G_1 \\ &= [1 \ -2] \begin{bmatrix} s & -1 \\ -2\alpha & s-\alpha+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1 \ -2] \frac{1}{s(s-\alpha+2)-2\alpha} \begin{bmatrix} s-\alpha+2 & 1 \\ 2\alpha & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{-s+3-5\alpha}{s^2+(2-\alpha)s-2\alpha} \end{aligned}$$

**N.B.**

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ * & C^{-1} \end{bmatrix}$$

## Esercizio 3

$$\dot{x}(t) = Fx(t) = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} x(t)$$

$$y(t) = Hx(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(t)$$

1. Per $\alpha = \beta = 1$, $x(0) \neq 0$ tali che $y(t)$ non è divergente?
2. Per $\alpha = 1$ e $\beta = -1$, $x(0) \neq 0$ tali che $y(t)$ è limitata?



$$F = \left[\begin{array}{cc|c} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

1) $\underline{\alpha = \beta = 1}$

$$y(t) = H e^{Ft} x(0)$$

Calcolo di $e^{Ft}$: $\quad e^{Ft} = \begin{bmatrix} e^{F_{11}t} & 0 \\ 0 & e^{F_{22}t} \end{bmatrix} \longrightarrow e^{F_{11}t}$

Calcolo $e^{F_{11}t} = T e^{F_D t} T^{-1} \qquad T^{-1} F T = F_D$

(i) Autovalori di $F_{11}$: $\quad \Delta_{F_{11}}(\lambda) = \det(\lambda I - F_{11}) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1$

$$\left.\begin{array}{ll} \lambda_1 = 1 & \nu_1 = 1 = g_1 \\ \lambda_2 = -1 & \nu_2 = 1 = g_2 \end{array}\right\} F_{11} \text{ diagonalizzabile}$$

(ii) Autovettori di $F_{11}$:

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ autovettore di $\lambda_1 = 1$: $\quad (\lambda_1 I - F_{11}) v = 0 \qquad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} v_1 = v_2 \\ \quad " \end{cases} \qquad v = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ autovettore di $\lambda_2 = -1$: $\quad (\lambda_2 I - F_{11}) v = 0 \qquad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} v_1 = -v_2 \\ \quad " \end{cases} \qquad v = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$

Matrice di cambio base: $\quad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$e^{Ft} = \begin{bmatrix} e^{F_{11}t} & 0 \\ 0 & e^{F_{22}t} \end{bmatrix} = \begin{bmatrix} T e^{F_{11,0}t} T^{-1} & 0 \\ 0 & e^{2t} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$T e^{F_{11,0}t} T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}$$

$$y(t) = H e^{Ft} x(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} & 0 \\ e^{t}-e^{-t} & e^{t}+e^{-t} & 0 \\ 0 & 0 & 2e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$= e^{t} x_{0,1} + e^{t} x_{0,2} + e^{2t} x_{0,3} \qquad x(0) = \begin{bmatrix} \gamma \\ -\gamma \\ 0 \end{bmatrix} \qquad \gamma \in \mathbb{R}, \ \gamma \neq 0$$

$$= 0$$

2) $\alpha = 1$, $\beta = -1$

$$F = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right] \qquad e^{Ft} = \begin{bmatrix} e^{F_{11}t} & 0 \\ 0 & e^{F_{22}t} \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$y(t) = He^{Ft}x(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$= \begin{bmatrix} \cos t - \sin t & \cos t + \sin t & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{bmatrix}$$

$$x(0) = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{bmatrix} \qquad \gamma_1, \gamma_2 \in \mathbb{R}$$

(almeno uno tra $\gamma_1$ e $\gamma_2 \neq 0$)

**Esercizio 4** [riadattato da Es. 3 tema d'esame 24 Giugno 2019]

$$x(t+1) = Fx(t) + Gu(t) = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = x(t)$$

1. Modi del sistema e loro carattere?
2. Matrice di trasferimento $W(z)$?
3. Evoluzione del sistema per $x(0) = \begin{bmatrix} \frac{1}{10} \\ 0 \end{bmatrix}$ e $u(t) = \delta(t)$, $t \geq 0$?

note

G. Baggio    Lez. 8: Esercizi di ricapitolazione Parte I    12 Marzo 2021

$$F = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1) Modi e carattere.

Autovalori di F: $\lambda_1 = -3 \quad \nu_1 = 1 = g_1$

$\lambda_2 = 2 \quad \nu_2 = 1 = g_2$

Forma di Jordan: $F_J = \begin{bmatrix} -3 & \\ & 2 \end{bmatrix}$ modi: $(-3)^t$, $2^t$ divergenti

2) $W(z) = H(zI - F)^{-1} G$

$$= \begin{bmatrix} z+3 & -1 \\ 0 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(z+3)(z-2)} \begin{bmatrix} z-2 & 1 \\ 0 & z+3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(z+3)(z-2)} \begin{bmatrix} 1 \\ z+3 \end{bmatrix} = \begin{bmatrix} \frac{1}{(z+3)(z-2)} \\ \frac{1}{z-2} \end{bmatrix}$$

3) $x(0) = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} \quad u(t) = \delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

$$X(z) = \underbrace{z(zI - F)^{-1} x(0)}_{X_\ell(z)} + \underbrace{W(z)U(z)}_{X_f(z)}$$

$$X_\ell(z) = \frac{z}{(z+3)(z-2)} \begin{bmatrix} z-2 & 1 \\ 0 & z+3 \end{bmatrix} \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \frac{z}{(z+3)\cancel{(z-2)}} \begin{bmatrix} \frac{1}{10}\cancel{(z-2)} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \frac{z}{z+3} \\ 0 \end{bmatrix} \xrightarrow{\mathcal{Z}^{-1}} \frac{1}{10}(-3)^t$$

$$X_f(z) = W(z)\underbrace{U(z)}_{=1} = \begin{bmatrix} \frac{1}{(z+3)(z-2)} \\ \frac{1}{z-2} \end{bmatrix} = \begin{bmatrix} X_{f,1}(z) \\ X_{f,2}(z) \end{bmatrix}$$

Calcolo di $x_{f,1}(t)$:

$$X'_{f,1}(z) = \frac{X_{f,1}(z)}{z} = \frac{1}{z(z+3)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z+3} + \frac{A_3}{z-2}$$

$$A_1 = -1/6, \quad A_2 = 1/15, \quad A_3 = 1/10$$

$$X_{f,1}(z) = z\,X'_{f,1}(z) = A_1 + \frac{A_2 z}{z+3} + \frac{A_3 z}{z-2}$$

$$\downarrow \mathcal{Z}^{-1} \qquad \downarrow \mathcal{Z}^{-1} \qquad \downarrow \mathcal{Z}^{-1}$$

$$A_1\delta(t) \qquad A_2(-3)^t \qquad A_3\, 2^t$$

Calcolo di $x_{f,2}(t)$:

$$X'_{f,2}(z) = \frac{X_{f,2}(z)}{z} = \frac{1}{z(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-2} \qquad A_1 = -1/2, \quad A_2 = 1/2$$

$$X_{f,2} = z\,X'_{f,2}(z) = A_1 + \frac{A_2 z}{z-2}$$

$$\downarrow \mathcal{Z}^{-1} \qquad \downarrow \mathcal{Z}^{-1}$$

$$A_1\delta(t) \qquad A_2\, 2^t$$