Modi elementari: osservazioni

$$e^{\lambda_i t},\ te^{\lambda_i t},\ \frac{t^2}{2}e^{\lambda_i t},\ \dots,\ \frac{t^{r_{ij}-1}}{(r_{ij}-1)!}e^{\lambda_i t} = \text{modi elementari del sistema}$$

1. Numero di modi *distinti* associati a λ_i = dim. del più grande miniblocco in J_{λ_i}
2. Numero di modi distinti complessivi $= n$ (dim. di F)
   solo quando F ha un solo miniblocco per ogni autovalore !
3. F diagonalizzabile $\Longrightarrow$ modi elementari $= e^{\lambda_i t}$ (esponenziali puri)
4. $\lambda \in \mathbb{C}$ autovalore $\Rightarrow \bar{\lambda}$ autovalore $\Rightarrow$ modi reali $t^k e^{\sigma t}\cos(\omega t)$, $t^k e^{\sigma t}\sin(\omega t)$

note

$F \in \mathbb{R}^{n\times n}$: $\lambda \in \mathbb{C}$ è autovalore $\Rightarrow$ $\bar{\lambda} = \sigma - i\omega$ è autovalore

$\lambda = \sigma + i\omega$

$$[e^{Ft}]_{ij} = c\,e^{\lambda t} + \bar{c}\,e^{\bar{\lambda} t} \qquad c = a+ib$$

$$= (a+ib)\,e^{(\sigma+i\omega)t} + (a-ib)\,e^{(\sigma-i\omega)t}$$

$$= (a+ib)\,e^{\sigma t}(\cos(\omega t) + i\sin(\omega t)) + (a-ib)\,e^{\sigma t}(\cos(\omega t) - i\sin(\omega t))$$

$$= 2a\,e^{\sigma t}\cos(\omega t) - 2b\,e^{\sigma t}\sin(\omega t)$$

Evoluzione forzata

$$\dot{x}(t) = Fx(t) + Gu(t), \qquad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = x_\ell(t) + x_f(t), \qquad x_\ell(t) = e^{Ft}x_0, \qquad x_f(t)\ ??$$

$$y(t) = y_\ell(t) + y_f(t), \qquad y_\ell(t) = He^{Ft}x_0, \qquad y_f(t)\ ??$$

note

$$\begin{cases} \dot{x}(t) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Ju(t) \end{cases}$$

Osservazioni:

1) $(e^{Ft})^{-1} = e^{-Ft}$ ($\forall F$ e^{Ft} è sempre invertibile)

2) $$\frac{d}{dt} e^{Ft} = \frac{d}{dt}\left(\sum_{K=1}^{\infty} \frac{1}{K!} F^k t^k\right) = \frac{d}{dt}\left(I + Ft + \frac{F^2}{2}t^2 + \frac{F^3}{3!}t^3 + \frac{F^4}{4!}t^4 + \dots\right)$$

$$= F + F^2 t + \frac{\cancel{3}F^3t^2}{\cancel{3}\cdot 2} + \frac{\cancel{4}F^4t^3}{\cancel{4}\cdot 3!} + \dots$$

$$= F\left(I + Ft + \frac{F^2}{2}t^2 + \frac{F^3}{3!}t^3 + \dots\right)$$

$$= F e^{Ft} = e^{Ft} F$$

$$\dot{x}(t) = Fx(t) + Gu(t)$$

$$e^{-Ft}\dot{x}(t) = e^{-Ft}Fx(t) + e^{-Ft}Gu(t)$$

$$\boxed{e^{-Ft}\dot{x}(t) - e^{-Ft}Fx(t)} = e^{-Ft}Gu(t)$$

$$\frac{d}{dt}\left(e^{-Ft}x(t)\right) = -e^{-Ft}Fx(t) + e^{-Ft}\dot{x}(t)$$

$$\frac{d}{dt}\left(e^{-Ft}x(t)\right) = e^{-Ft}Gu(t)$$

$$\int_0^t \frac{d}{d\tau}\left(e^{-F\tau}x(\tau)\right)d\tau = \int_0^t e^{-F\tau}Gu(\tau)\,d\tau$$

$$e^{-Ft}x(t) - e^{-F\cdot 0}x(0) = \int_0^t e^{-F\tau}Gu(\tau)\,d\tau$$

$$e^{-Ft}x(t) - \overbrace{e^{-F\cdot 0}}^{I}x(0) = \int_0^t e^{-F\tau} G u(\tau)\, d\tau$$

$$x(t) = \underbrace{e^{Ft}x(0)}_{x_\ell(t)} + \underbrace{\int_0^t e^{F(t-\tau)} G u(\tau)\, d\tau}_{x_f(t)}$$

$$y(t) = Hx(t) + Ju(t)$$

$$= He^{Ft}x(0) + \int_0^t He^{F(t-\tau)} G u(\tau)\, d\tau + Ju(t)$$

delta di Dirac

$$= \underbrace{He^{Ft}x(0)}_{y_\ell(t)} + \underbrace{\int_0^t \left[He^{F(t-\tau)}G + J\delta(t-\tau)\right] u(\tau)\, d\tau}_{y_f(t)}$$

$w(t) = He^{Ft}G + J\delta(t)$ = risposta impulsiva

$$y_f(t) = [w * u](t)$$

prodotto di convoluzione

Evoluzione forzata (con Laplace)

$sX(s) - x_0 = FX(s) + GU(s)$

$Y(s) = HX(s) + JU(s)$

$V(s) \triangleq \mathcal{L}[v(t)] = \int_{0^-}^{\infty} v(t)e^{-st}\,\mathrm{d}t$

note

Trasformata di Laplace:

$$V(s) = \mathcal{L}[v(t)] = \int_{0^-}^{\infty} v(t)\, e^{-st}\, dt$$

$$\mathcal{L}[\dot{v}(t)] = s\,V(s) - v(0^-)$$

$$\begin{cases} \dot{x}(t) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Ju(t) \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sX(s) - x(0) = F\,X(s) + GU(s) \\ Y(s) = H\,X(s) + J\,U(s) \end{cases}$$

$$\begin{cases} (sI - F)\,X(s) = x(0) + GU(s) \\ \end{cases}$$

$$\begin{cases} X(s) = \overbrace{(sI-F)^{-1}x(0)}^{X_\ell(s)} + \overbrace{(sI-F)^{-1}GU(s)}^{X_f(s)} \\ Y(s) = \underbrace{H(sI-F)^{-1}x(0)}_{Y_\ell(s)} + \underbrace{[H(sI-F)^{-1}G + J]\,U(s)}_{Y_f(s)} \end{cases}$$

$$W(s) = \frac{Y(s)}{U(s)}$$

1) $W(s) = \mathcal{L}[w(t)] = H(sI-F)^{-1}G + J$ = matrice di trasferimento

2) $X_\ell(s) = \mathcal{L}[x_\ell(t)] = \mathcal{L}[e^{Ft}x(0)] = \mathcal{L}[e^{Ft}]\,x(0)$

$= (sI - F)^{-1}x(0)$

$$\Longrightarrow \boxed{\mathcal{L}[e^{Ft}] = (sI - F)^{-1}}$$

Equivalenza algebrica

$$\dot{x}(t) = Fx(t) + Gu(t), \qquad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

Sia $z \triangleq T^{-1}x$ dove $T \in \mathbb{R}^{n\times n}$ rappresenta una matrice di cambio di base

Equazioni del sistema espresse nella nuova base?

$$\begin{cases} \dot{x}(t) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Ju(t) \end{cases} \qquad x(0) = x_0$$

$$z(t) = T^{-1}x(t) \implies x(t) = Tz(t)$$

$$\dot{x}(t) = T\dot{z}(t)$$

$$\begin{cases} T\dot{z}(t) = FTz(t) + Gu(t) \\ y(t) = HTz(t) + Ju(t) \end{cases} \qquad \begin{cases} \dot{z}(t) = T^{-1}FTz(t) + T^{-1}Gu(t) \\ y(t) = HTz(t) + Ju(t) \end{cases} \qquad z(0) = T^{-1}x(0)$$

$$\Sigma = (F, G, H, J) \xrightarrow{z = T^{-1}x} \Sigma' = (T^{-1}FT, T^{-1}G, HT, J)$$

Σ, Σ' algebricamente equivalenti

Equivalenza algebrica

$$\dot{z}(t) = T^{-1}FTz(t) + T^{-1}Gu(t), \qquad z(0) = Tx_0$$

$$y(t) = HTz(t) + Ju(t)$$

$$(F, G, H, J) \xrightarrow{z=T^{-1}x} (F' = T^{-1}FT,\ G' = T^{-1}G,\ H' = HT,\ J' = J)$$

Matrice di trasferimento nella nuova base?

note

$$\Sigma = (F, G, H, J)$$

$$\Sigma' = (T^{-1}FT, T^{-1}G, HT, J)$$

$$W(s) = H(sI - F)^{-1}G + J$$

$$W'(s) = HT(sI - T^{-1}FT)^{-1}T^{-1}G + J$$

$$\overset{!}{=} HT(T^{-1}(sI - F)T)^{-1}T^{-1}G + J$$

$$\overset{!}{=} H\cancel{T}\cancel{T^{-1}}(sI - F)^{-1}\cancel{T}\cancel{T^{-1}}G + J = W(s)$$

Struttura della matrice di trasferimento

miniblocco $J_{\lambda_i,j} \in \mathbb{R}^{r_{ij}\times r_{ij}} \implies W_{\lambda_i,j}(s) = \frac{A_1}{s-\lambda_i} + \frac{A_2}{(s-\lambda_i)^2} + \cdots + \frac{A_{r_{ij}}}{(s-\lambda_i)^{r_{ij}}}$

$$y_f(t) = \mathcal{L}^{-1}\left[\sum_{i,j} W_{\lambda_i,j}(s)U(s) + JU(s)\right]$$

note

$$W_{\lambda_{i,j}}(s) = H_{\lambda_{i,j}}\left(sI - J_{\lambda_{i,j}}\right)^{-1} G_{\lambda_{i,j}}$$

$$J_{\lambda_{i,j}} = \begin{bmatrix} \lambda_i & 1 & \\ & \lambda_i & 1 \\ & & \lambda_i \end{bmatrix} = \lambda_i I + \overbrace{\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}}^{N}$$

$$\left(sI - J_{\lambda_{i,j}}\right)^{-1} = \left((s-\lambda_i)I - N\right)^{-1}$$

$$L = \frac{I}{s-\lambda_i} + \frac{N}{(s-\lambda_i)^2} + \frac{N^2}{(s-\lambda_i)^3}$$

$$\left(sI - J_{\lambda_{i,j}}\right)L = \left((s-\lambda_i)I - N\right)\left(\frac{I}{s-\lambda_i} + \frac{N}{(s-\lambda_i)^2} + \frac{N^2}{(s-\lambda_i)^3}\right)$$

$$= I - \cancel{\frac{N}{s-\lambda_i}} + \cancel{\frac{N}{s-\lambda_i}} - \cancel{\frac{N^2}{(s-\lambda_i)^2}} + \cancel{\frac{N^2}{(s-\lambda_i)^2}} - \cancel{\frac{N^3}{(s-\lambda_i)^3}} \qquad N^3 = 0$$

$$= I$$

$$\underline{\left(sI - J_{\lambda_{i,j}}\right)^{-1} = L}$$

$$\left(sI - J_{\lambda_{i,j}}\right)^{-1} = \frac{I}{s-\lambda_i} + \frac{N}{(s-\lambda_i)^2} + \cdots + \frac{N^{r_{ij}-1}}{(s-\lambda_i)^{r_{ij}}}$$

$$W_{\lambda_{i,j}}(s) = H_{\lambda_{i,j}}\left(sI - J_{\lambda_{i,j}}\right)^{-1} G_{\lambda_{i,j}} = \frac{H_{\lambda_{i,j}} G_{\lambda_{i,j}}}{s-\lambda_i} + \cdots + \frac{H_{\lambda_{i,j}} N^{r_{ij}-1} G_{\lambda_{i,j}}}{(s-\lambda_i)^{r_{ij}}}$$