Esempio: diagonalizzazione

$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, F diagonalizzabile? Se sì, calcolare T.

$\lambda_1 = i$, $\nu_1 = 1$, $g_1 = 1$, $\lambda_2 = -i$, $\nu_2 = 1$, $g_2 = 1 \implies F$ diagonalizzabile ✓

$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \implies F_D = T^{-1}FT = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ F diagonalizzabile?

Calcolo autovalori:

$\Delta_F(\lambda) = \det(\lambda I - F) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 1 \qquad \lambda_{1,2} = \pm i$

$$\left.\begin{array}{lll} \lambda_1 = +i & \nu_1 = 1 = g_1 & 1 \leq g_i \leq \nu_i \\ \lambda_2 = -i & \nu_2 = 1 = g_2 & \end{array}\right\} \Rightarrow F \text{ è diagonalizzabile}$$

Calcolo autovettori:

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ autovettore relativo a $\lambda_1 = +i$:

$(\lambda_1 I - F)v = 0 \qquad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} iv_1 - v_2 = 0 \\ v_1 + iv_2 = 0 \end{cases} \qquad \begin{cases} v_2 = iv_1 \\ \text{"} \end{cases} \rightarrow v = \begin{bmatrix} \alpha \\ i\alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ autovettore relativo a $\lambda_2 = -i$:

$(\lambda_2 I - F)v = 0 \qquad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} -iv_1 - v_2 = 0 \\ v_1 - iv_2 = 0 \end{cases} \qquad \begin{cases} v_2 = -iv_1 \\ \text{"} \end{cases} \rightarrow v = \begin{bmatrix} \alpha \\ -i\alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$

Matrice di cambio base: $\underset{\alpha = 1}{T} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad T^{-1}FT = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Calcolo di e^{Ft} tramite diagonalizzazione: esempio

$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, calcolare e^{Ft} tramite diagonalizzazione di F.

$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, $F_D = T^{-1}FT = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$e^{Ft} = Te^{F_D t}T^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

note

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$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad F_D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Inversa matrice 2x2:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} = \frac{1}{-2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$e^{Ft} = e^{TF_DT^{-1}t} = T e^{F_D t} T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{it}+e^{-it} & -ie^{it}+ie^{-it} \\ ie^{it}-ie^{-it} & e^{it}+e^{-it} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{it}+e^{-it}}{2} & \frac{1}{2i}\left(e^{it}-e^{-it}\right) \\ \frac{1}{2i}\left(-e^{it}+e^{-it}\right) & \frac{e^{it}+e^{-it}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Formule di Eulero:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

Esempi

1. $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

note

1) $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \nu_1 = 2 = g_1 \quad$ diagonalizzabile

2) $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \Delta_F(\lambda) = \det(\lambda I - F) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix}$

$= (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda + \cancel{1} - \cancel{1}$

$= \lambda(\lambda - 2)$

$\left.\begin{matrix} \lambda_1 = 0 & \nu_1 = 1 = g_1 \\ \lambda_2 = 2 & \nu_2 = 1 = g_2 \end{matrix}\right\} \Rightarrow$ diagonalizzabile

3) $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \nu_1 = 2$

$g_1 = 2 - \text{rank}(\lambda_1 I - F) = 2 - \underbrace{\text{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{1} = 2 - 1 = 1$

$\nu_1 > g_1 \Rightarrow$ non diagonalizzabile

Forma di Jordan: esempi

1. $F = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. $F = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \alpha = 0, 1$

note

1) $F = \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}$

$\lambda(F) = \lambda(F_{11}) \cup \lambda(F_{22})$

$\Delta_{F_{11}}(\lambda) = \det(\lambda I - F_{11}) = \det \begin{bmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 1 \end{bmatrix} = (\lambda-3)(\lambda-1) + 1$
$= \lambda^2 - 4\lambda + 4$
$= (\lambda - 2)^2$

Autovalori di F: $\lambda_1 = 2 \quad \nu_1 = 3$

$g_1 = 3 - \text{rank}(\lambda_1 I - F) = 3 - \text{rank} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2$

$F_J \longrightarrow g_1 = 2$ miniblocchi relativi a $\lambda_1 = 2$ → 1 miniblocco 2x2
→ 1 miniblocco 1x1

$F_J = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$

2) $F = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\alpha = 0, 1$

$\lambda_1 = 1 \qquad \nu_1 = 4$

$$g_1 = 4 - \text{rank}(I - F) = 4 - \underbrace{\text{rank}\begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{2} = 2$$

$F_J \to g_1 = 2$ miniblocchi relativi a $\lambda_1 = 1$

→ 2 miniblocchi 2×2

→ 1 miniblocco 3×3
1 miniblocco 1×1

$$F_J = \begin{cases} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \alpha = 0 \\[2ex] \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \alpha = 1 \end{cases}$$