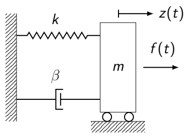
Massa-molla-smorzatore



$f(t) = \text{input}$, $z(t) = \text{output}$

Rappresentazione (esterna ed) interna?

note

$z(t)$ = posizione del carrello al tempo t = output

$f(t)$ = forza esterna al tempo t = input

Rappresentazione esterna:

$$m\ddot{z} = f(t) - Kz - \beta\dot{z}$$

$$m\ddot{z} + \beta\dot{z} + Kz - f(t) = 0$$

dominio Laplace

$$m s^2 Z(s) + \beta s Z(s) + k Z(s) = F(s)$$

$$G(s) = \frac{Z(s)}{F(s)} = \frac{1}{ms^2 + \beta s + k}$$

Rappresentazione interna:

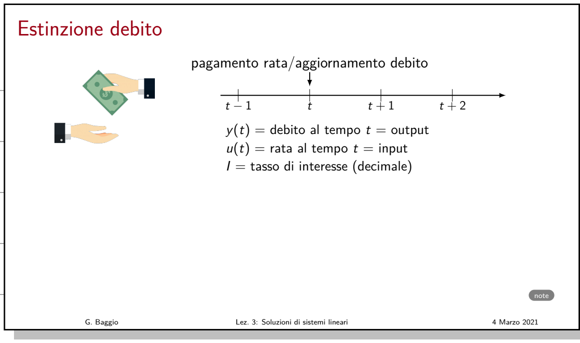
$$x(t) = \begin{cases} \text{posizioni} \\ \text{velocità} \end{cases} \text{ delle masse}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad x_1(t) = z(t), \quad x_2(t) = \dot{z}(t) \qquad u(t) = f(t)$$

$$\dot{x}_1 = \dot{z} = x_2$$

$$\dot{x}_2 = \ddot{z} = \frac{1}{m}\left(-\beta\dot{z} - kz + f\right) = \frac{1}{m}\left(-\beta x_2 - kx_1 + u\right)$$

$$\begin{cases} \dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix}}^{F} x + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{G} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} x + \underbrace{0}_{J} \cdot u \end{cases}$$



$y(t) =$ debito al tempo $t =$ output

$u(t) =$ rata al tempo $t =$ input

Rappresentazione esterna:

$$y(t+1) = (1+I)\,y(t) - u(t+1)$$

$$y(t+1) - (1+I)\,y(t) + u(t+1) = 0$$

$\hookrightarrow$ trasformate zeta

$$z\,Y(z) - (1+I)\,Y(z) = z\,U(z) \implies G(z) = \frac{z}{z-(1+I)} \qquad \text{F.d.T.}$$

Rappresentazione interna:

$$x(t) = y(t) + u(t)$$

$$x(t+1) = y(t+1) + u(t+1) = (1+I)\,y(t) - \cancel{u(t+1)} + \cancel{u(t+1)}$$
$$= (1+I)\,(x(t) - u(t))$$
$$= (1+I)\,x(t) - (1+I)\,u(t)$$

$$y(t) = x(t) - u(t)$$

$$\begin{cases} x(t+1) = \overbrace{(1+I)}^{F}\,x(t) - \overbrace{(1+I)}^{G}\,u(t) \\ y(t) = \underbrace{1}_{H}\cdot x(t) \underbrace{-}_{J} u(t) \end{cases}$$

Esponenziale di matrice e sue proprietà

Definizione: L'esponenziale di una matrice $A \in \mathbb{R}^{n\times n}$ è definito come

$$e^A \triangleq \sum_{k\geq 0} \frac{A^k}{k!}.$$

NB: e^A è sempre ben definito perché la serie $\sum_{k\geq 0} \frac{A^k}{k!}$ converge sempre!

(Alcune) proprietà:

1. $e^0 = I$
2. $(e^A)^\top = e^{(A^\top)}$
3. $AB = BA \implies e^{A+B} = e^A e^B$
4. $T \in \mathbb{R}^{n\times n}$ invertibile: $e^{TAT^{-1}} = Te^AT^{-1}$

note

1) $e^0 = I$

$$e^0 = \sum_{k=0}^{\infty} \frac{1}{k!} 0^k = \underbrace{\frac{0^0}{0!}}_{I} + \underbrace{\frac{0^1}{1!} + \frac{0^2}{2!} + \dots}_{0}$$

2) $(e^A)^T = e^{A^T}$

$$(e^A)^T = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)^T = \sum_{k=0}^{\infty} \frac{(A^k)^T}{k!} = \sum_{k=0}^{\infty} \frac{(A^T)^k}{k!} = e^{A^T}$$

$$(A \cdots A)^T = A^T \cdots A^T$$

3) $A, B \in \mathbb{R}^{n\times n}$: $AB = BA$ (A, B commutano)

$\left[A, B \text{ commutano: } (A+B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \right]$

$$e^{A+B} = e^A e^B$$

Corollari: 1) $\alpha A, \beta A$ $\alpha, \beta \in \mathbb{R}$: $e^{\alpha A + \beta A} = e^{\alpha A} e^{\beta A}$

2) $A, -A$: $e^{A-A} = e^A e^{-A} = e^0 = I$

$$(e^A)^{-1} = e^{-A}$$

4) $T \in \mathbb{R}^{n\times n}$ invertibile:

$$e^{TAT^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!} (TAT^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} TA^kT^{-1} = T\left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k\right)T^{-1} = Te^AT^{-1}$$

$$(TAT^{-1})^2 = TA\cancel{T^{-1}}\cancel{T}AT^{-1} = TA^2T^{-1}$$

$$\vdots$$

$$(TAT^{-1})^k = TA^kT^{-1}$$

Come calcolare e^{Ft}, $F \in \mathbb{R}^{n\times n}$?

Usiamo la definizione: $e^{Ft} \triangleq \sum_{k\geq 0} \frac{F^k t^k}{k!}$

Esempio 1: $F = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

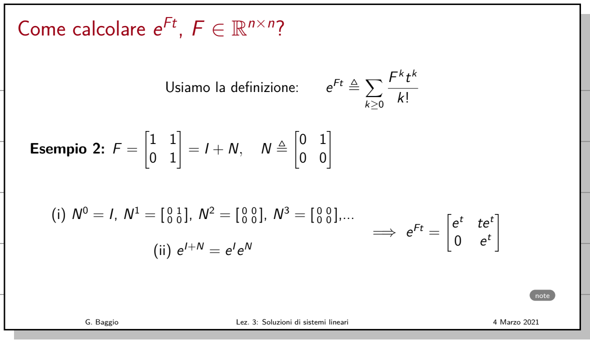
$$(Ft)^n = \underbrace{F \cdot F \cdots F}_{n \text{ volte}} t^n = \begin{bmatrix} t^n & 0 \\ 0 & (2t)^n \end{bmatrix} \implies e^{Ft} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

note

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{1}{k!} (Ft)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} t^k & 0 \\ 0 & 2^k t^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} t^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} (2t)^k \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$



$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N}$$

Osservazione: I, N commutano $\quad IN = NI$

$$e^{Ft} = e^{(N+I)t} = e^{It} e^{Nt} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

$$e^{It} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$$

$$e^{Nt} = \sum_{k=0}^{\infty} \frac{1}{k!} N^k t^k = I + Nt + \frac{N^2 t^2}{2} + \dots$$

$$N^2 = NN = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\vdots$

$$N^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad k \geq 2$$

$$e^{Nt} = I + Nt = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Definizione: $N \in \mathbb{R}^{n \times n}$ è detta nilpotente se $\exists\, \bar{k} \geq 1$ t.c.

$$N^{\bar{k}} = 0 \quad k \geq \bar{k}$$

Indice di nilpotenza è il più piccolo valore di $\bar{k}$

Come calcolare e^{Ft}, $F \in \mathbb{R}^{n\times n}$?

Usiamo la definizione: $e^{Ft} \triangleq \sum_{k\geq 0} \frac{F^k t^k}{k!}$

Esempio 3: $F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = I + N, \quad N \triangleq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(i) $N^0 = I$, $N^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots$ $\implies e^{Ft} = \begin{bmatrix} e^t & te^t & \frac{t^2}{2!}e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$

(ii) $e^{I+N} = e^I e^N$

note

$$F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = I + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

Osservazione: I, N commutano

$$e^{Ft} = e^{(I+N)t} = e^{It} e^{Nt} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t & \frac{t^2}{2} e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

$$e^{Nt} = \sum_{k=0}^{\infty} \frac{1}{k!} N^k t^k = I + Nt + \frac{N^2}{2} t^2 + \frac{N^3}{3!} t^3 + \dots$$

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = N^2 N = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^k = 0 \quad k \geq 3$$

$$e^{Nt} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Come calcolare e^{Ft}, $F \in \mathbb{R}^{n\times n}$?

Usiamo la definizione: $e^{Ft} \triangleq \sum_{k\geq 0} \frac{F^k t^k}{k!}$

Esempio 4: $F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$F^0 = I,\ F^1 = F,\ F^2 = -I,\ F^3 = -F,\ F^4 = I, \ldots \implies e^{Ft} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

note

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e^{Ft}\ ?$$

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{1}{k!} F^k t^k = I + Ft + \frac{F^2}{2} t^2 + \frac{F^3}{3!} t^3 + \frac{F^4}{4!} t^4 + \frac{F^5 t^5}{5!} + \ldots$$

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$F^2 = FF = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$F^3 = F^2 F = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -F$$

$$F^4 = F^3 F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$F^5 = F^4 F = F$$

$$e^{Ft} = \begin{bmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \ldots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \ldots & 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \ldots \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Come calcolare e^{Ft}, $F \in \mathbb{R}^{n\times n}$?

Usiamo la definizione: $e^{Ft} \triangleq \sum_{k\geq 0} \frac{F^k t^k}{k!}$

Esempio 5: $F = F^2$

$$F^0 = I,\ F^k = F,\ k \geq 1 \implies e^{Ft} = I + (e^t - 1)F$$

note

$F^2 = F, \quad F \in \mathbb{R}^{n\times n}$

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{1}{k!} F^k t^k = I + \sum_{k=1}^{\infty} \frac{1}{k!} F^k t^k = I + F \underbrace{\sum_{k=1}^{\infty} \frac{t^k}{k!}}_{\underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!}}_{e^t} - 1}$$

$F^2 = F$

$F^3 = F^2 F = F^2 = F$

$\vdots$

$F^k = F \quad k \geq 1$

$$= I + F(e^t - 1)$$

Come calcolare e^{Ft}, $F \in \mathbb{R}^{n\times n}$?

Usiamo la definizione: $e^{Ft} \triangleq \sum_{k\geq 0} \frac{F^k t^k}{k!}$

Esempio 6: $F = vu^\top$, $v, u \in \mathbb{R}^n$

$$F^0 = I,\ F^k = (u^\top v)F^{k-1},\ k \geq 1 \implies e^{Ft} = I + \frac{(e^{u^\top v t} - 1)}{u^\top v} F$$

note

$F = vu^T \quad v, u \in \mathbb{R}^n \quad$ rango 1

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{1}{k!} F^k t^k = I + \sum_{k=1}^{\infty} \frac{1}{k!} F^k t^k = I + \sum_{k=1}^{\infty} \frac{1}{k!} (u^T v)^{k-1} F t^k$$

$$F = vu^T$$

$$F^2 = v(u^T v)u^T = (u^T v) vu^T = (u^T v) F$$

$$F^3 = F^2 F = (u^T v) v(u^T v)u^T = (u^T v)^2 vu^T = (u^T v)^2 F$$

$\vdots$

$$F^k = (u^T v)^{k-1} F \qquad k \geq 1$$

$$= I + \frac{F}{(u^T v)} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k!} (u^T v)^k t^k}_{\sum_{k=0}^{\infty} \frac{1}{k!} (u^T v t)^k - 1}$$

$$= I + \frac{F}{u^T v} \left(e^{u^T v t} - 1\right)$$