Esercizio 1

$x(t+1) = Fx(t)$, $\quad F = \begin{bmatrix} 1 & 1 & \alpha - \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix}$, $\quad \alpha \in \mathbb{R}$

$y(t) = Hx(t)$, $\quad H = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$

1. Osservabilità, ricostruibilità e rivelabilità al variare di $\alpha \in \mathbb{R}$?

2. Spazi non osservabili $X_{NO}(t)$, $t \geq 1$, al variare di $\alpha \in \mathbb{R}$?

note

$$F = \begin{bmatrix} 1 & 1 & \alpha - 1/2 \\ 0 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

$$H = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

1) Osservabilità, ricostruibilità, rivelabilità per $\alpha \in \mathbb{R}$.

Autovalori di F: $1, \alpha$

<u>Caso $\alpha = 1$</u>: $\lambda_1 = 1$, $\nu_1 = 3$

Test PBH di osservabilità:

$$PBH(\lambda_1) = \begin{bmatrix} \lambda_1 I - F \\ H \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1/2 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \text{rank}(PBH(\lambda_1)) = 3$$

$\Rightarrow \Sigma$ osservabile

$\Rightarrow \Sigma$ ricostr., rivelabile

<u>Caso $\alpha \neq 1$</u>: $\lambda_1 = 1$, $\nu_1 = 2$, $\lambda_2 = \alpha$, $\nu_1 = 1$

$$PBH(\lambda_1) = \begin{bmatrix} \lambda_1 I - F \\ H \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1/2 - \alpha \\ 0 & 0 & 0 \\ 0 & -1 & 1 - \alpha \\ 1 & 1 & 0 \end{bmatrix} \qquad \text{rank}(PBH(\lambda_1)) = 3 \quad \forall \alpha$$

$$PBH(\lambda_2) = \begin{bmatrix} \lambda_2 I - F \\ H \end{bmatrix} = \begin{bmatrix} \alpha - 1 & -1 & 1/2 - \alpha \\ 0 & \alpha - 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \text{rank}(PBH(\lambda_2)) = \begin{cases} 2 & \text{se } \alpha = 1/2 \\ 3 & \text{se } \alpha \neq 1/2 \end{cases}$$

Σ osservabile se $\alpha \neq \frac{1}{2}$

Σ ricostruibile se $\alpha \neq \frac{1}{2}$

Σ rivelabile $\forall \alpha \in \mathbb{R}$ (perché : 1) Σ oss $\Rightarrow$ Σ rivelabile ($\alpha \neq 1/2$)

2) $\alpha = 1/2$ matrice PBH(λ_2) cade di rango, ma in questo caso $\lambda_2 = 1/2$ e $|\lambda_2| < 1$)

2) Spazi non osservabili $X_{NO}(t)$, $t \geq 1$

$$X_{NO}(1) = \ker \mathcal{O}_1 = \ker H = \ker \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \left\{ x \in \mathbb{R}^3 : Hx = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 = 0 \right\} = \left\{ \begin{bmatrix} \beta \\ -\beta \\ \gamma \end{bmatrix}, \ \beta, \gamma \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$X_{NO}(2) = \ker \mathcal{O}_2 = \ker \begin{bmatrix} H \\ HF \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix}$$

$$= \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$X_{no}(2) = \ker \mathcal{O}_2 = \ker \begin{bmatrix} H \\ HF \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix}$$

$$= \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 + (\alpha - 1/2) x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \begin{cases} \left\{ \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}, \gamma \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} & \alpha = 1/2 \\ \left\{ \begin{bmatrix} (\alpha - 1/2)\gamma \\ (1/2 - \alpha)\gamma \\ \gamma \end{bmatrix}, \gamma \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} \alpha - 1/2 \\ 1/2 - \alpha \\ 1 \end{bmatrix} \right\} & \alpha \neq 1/2 \end{cases}$$

$X_{NO}(3)$:

$\alpha = 1/2$: $X_{NO}(3) = \mathrm{Ker}\begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \mathrm{Ker}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$= \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

$= \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

$\begin{cases} x_1 = -x_2 \\ x_1 = -2x_2 \\ x_1 = -3x_2 \end{cases} \quad \begin{cases} x_1 = 0 \\ -x_2 = -2x_2 \Rightarrow x_2 = 0 \\ \text{"} \end{cases}$

$= \left\{\begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix}, y \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\alpha \neq 1/2$: $X_{NO}(3) = \{0\}$ perché Σ osservabile

$X_{NO}(t) = X_{NO}(3) \quad \forall t \geq 3$

Esercizio 2 [riadattato da Es. 3 tema d'esame 30 Gennaio 2015]

$$x(t+1) = Fx(t) + Gu(t), \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Hx(t), \qquad H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1. Per quali uscite y_1, y_2 esiste uno stimatore dead-beat?
2. Stimatore con errore di stima con modi solo convergenti o oscillatori usando y_2?
3. Regolatore dead-beat usando la sola uscita y_1?

note

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

1) Esistenza stimatore dead-beat per $\Sigma^{(1)} = (F, h_1)$, $\Sigma^{(2)} = (F, h_2)$.

$\exists$ stimatore dead-beat per $\Sigma \iff \Sigma$ è ricostruibile

Test PBH applicato a $\Sigma^{(1)}$:

- Autovalori F: $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$

$$\bullet\ PBH(\lambda_1) = \begin{bmatrix} \lambda_1 I - F \\ h_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \implies \operatorname{rank} PBH(\lambda_1) = 3$$

$$\bullet\ PBH(\lambda_2) = \begin{bmatrix} \lambda_2 I - F \\ h_1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \implies \operatorname{rank} PBH(\lambda_2) = 3$$

$\Sigma^{(1)}$ ricostruibile

$\Downarrow$

$\exists$ stimatore dead-beat per $\Sigma^{(1)}$

Test PBH applicato a $\Sigma^{(2)}$:

$$\bullet\ PBH(\lambda_1) = \begin{bmatrix} \lambda_1 I - F \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \implies \operatorname{rank} PBH(\lambda_1) = 2$$

$\implies \Sigma^{(2)}$ non ricostruibile

$\implies \nexists$ stimatore dead-beat per $\Sigma^{(2)}$

2) Stimatore per $\Sigma^{(2)}$ tale che l'errore di stima contenga modi conv. o oscill.

Autovalori di F: $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$

Modi di $\Sigma^{(2)}$: 1, $(-1)^t$, $\delta(t)$

$\lambda_1 = 1$ è autovalore non osservabile di $\Sigma^{(2)}$

$\Longrightarrow$ $\forall L$, $F+LH$ avrà sempre un autovalore in $\lambda_1 = 1$

$\Longrightarrow$ l'errore di stima avrà sempre il modo 1

$\Longrightarrow$ lo stimatore richiesto non esiste!

3) Regolatore dead-beat per $\Sigma = (F, G, h_1)$

i) $\exists$ regolatore dead-beat $\Longleftrightarrow$ Σ controllabile e ricostruibile

$$R = [G \;\; FG \;\; F^2G] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Longrightarrow \text{rank}\, R = 3 \Longrightarrow \Sigma \text{ ragg.}$$

$$\Longrightarrow \Sigma \text{ contr.}$$

$$(\det R = -1+1-1-1 = -2 \neq 0)$$

Σ contr. e ricostruibile $\Longrightarrow$ $\exists$ regolatore dead-beat

ii) Calcolo regolatore dead-beat

- Calcolo K^* t.c. $\Delta_{F+GK^*}(\lambda) = \lambda^3$

$K = [k_1 \;\; k_2 \;\; k_3]$

$$\Delta_{F+GK}(\lambda) = \det(\lambda I - F - GK) = \det\begin{bmatrix} \lambda-1-k_1 & -k_2 & -k_3 \\ -k_1 & \lambda+1-k_2 & -k_3 \\ -1 & 0 & \lambda \end{bmatrix}$$

$$= \lambda(\lambda-1-k_1)(\lambda+1-k_2) - k_2k_3 - k_3(\lambda+1-k_2) - \lambda k_1 k_2$$

$$= \lambda\left(\lambda^2 + (-\cancel{1}-k_1\cancel{+1}-k_2)\lambda + (-1-k_1)(1-k_2)\right) - k_2k_3 - \lambda k_3 - k_3(1-k_2) - \lambda k_1 k_2$$

$$= \lambda^3 + (-k_1-k_2)\lambda^2 + \lambda(-1+k_2-k_1+\cancel{k_1k_2}) - \cancel{k_2k_3} - \lambda k_3 - k_3 + \cancel{k_2k_3} - \cancel{\lambda k_1 k_2}$$

$$= \lambda^3 + (-k_1-k_2)\lambda^2 + (-1+k_2-k_1-k_3)\lambda - k_3 \overset{!}{=} \lambda^3$$

$$\begin{cases} -k_1 - k_2 = 0 \\ -1 + k_2 - k_1 - k_3 = 0 \\ -k_3 = 0 \end{cases} \quad \begin{cases} k_1 = -k_2 \\ -1 + 2k_2 = 0 \\ k_3 = 0 \end{cases} \quad \begin{cases} k_1 = -1/2 \\ k_2 = 1/2 \\ k_3 = 0 \end{cases} \qquad K^* = \begin{bmatrix} -1/2 & 1/2 & 0 \end{bmatrix}$$

- Calcolo L^* t.c. $\Delta_{F+L^*h_1}(\lambda) = \lambda^3$

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

$$\Delta_{F+Lh_1}(\lambda) = \det(\lambda I - F - Lh_1) = \det\left[\begin{array}{cc:c} \lambda-1-l_1 & -l_1 & 0 \\ -l_2 & \lambda+1-l_2 & 0 \\ \hdashline -1+l_3 & -l_3 & \lambda \end{array}\right]$$

$$= \lambda \det\begin{bmatrix} \lambda-1-l_1 & -l_1 \\ -l_2 & \lambda+1-l_2 \end{bmatrix}$$

$$= \lambda \left((\lambda - 1 - l_1)(\lambda + 1 - l_2) - l_1 l_2 \right)$$

$$= \lambda \left(\lambda^2 + (-1 - l_1 + 1 - l_2)\lambda + (-1 - l_1)(1 - l_2) - l_1 l_2 \right)$$

$$= \lambda^3 + (-l_1 - l_2)\lambda^2 + \lambda\left(-1 - l_1 + l_2 + \cancel{l_1 l_2} - \cancel{l_1 l_2}\right)$$

$$\overset{!}{=} \lambda^3$$

$$\begin{cases} -l_1 - l_2 = 0 \\ -1 - l_1 + l_2 = 0 \end{cases} \qquad \begin{cases} l_1 = -l_2 \\ -1 + 2l_2 = 0 \end{cases} \qquad \begin{cases} l_1 = -1/2 \\ l_2 = 1/2 \end{cases} \qquad L^* = \begin{bmatrix} -1/2 \\ 1/2 \\ \alpha \end{bmatrix} \qquad \alpha \in \mathbb{R}$$