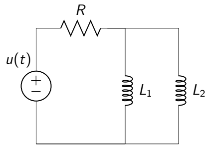
Esempio introduttivo



$x_1(t) = i_{L_1}(t),\ x_2(t) = i_{L_2}(t)$

$y(t) = i_R(t) = i_{L_1}(t) + i_{L_2}(t)$

$t_0 = 0,\ L_1 = L_2 = L$

$$x_1 = i_{L_1},\quad x_2 = i_{L_2}$$

$$y = i_{L_1} + i_{L_2} = x_1 + x_2$$

$$t_0 = 0,\quad L = L_1 = L_2$$

Stati non osservabili del sistema?

Rappresentazione in spazio di stato:

$$\dot{x}_1 = \frac{d i_{L_1}}{dt} = \frac{V_{L_1}}{L_1} = \frac{(u - V_R)}{L_1} = \frac{(u - R i_R)}{L_1} = \frac{1}{L_1}(u - R i_{L_1} - R i_{L_2})$$

$$= \frac{1}{L_1} u - \frac{R}{L_1} x_1 - \frac{R}{L_1} x_2$$

$$\dot{x}_2 = \frac{d i_{L_2}}{dt} = \frac{V_{L_2}}{L_2} = \frac{1}{L_2} u - \frac{R}{L_2} x_1 - \frac{R}{L_2} x_2$$

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix}}^{F} x + \overbrace{\begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_1} \end{bmatrix}}^{G} u \\ y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{H} x \end{cases}$$

$$L_1 = L_2 = L$$

$$y(t) = H e^{Ft} x_0 + \int_0^t H e^{F(t-\tau)} G u(\tau)\, d\tau \qquad x(0) = x_0$$

$$y_0(t) = \int_0^t H e^{F(t-\tau)} G u(\tau)\, d\tau$$

Se x_0 è non osservabile in $[0,t]$: $y(\tau) = y_0(\tau)$ $\tau \in [0,t]$ $\forall u(\tau)$

$$\Longrightarrow y(\tau) - y_0(\tau) = 0 \Longrightarrow H e^{F\tau} x_0 = 0 \quad \forall \tau \in [0,t]$$

Calcolo $e^{F\tau}$

$$F = -\frac{R}{L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = v u^T \qquad v = -\frac{R}{L}\begin{bmatrix}1\\1\end{bmatrix} \qquad u = \begin{bmatrix}1\\1\end{bmatrix}$$

$$e^{F\tau} = I + \frac{\left(e^{u^T v \tau} - 1\right)}{u^T v} F = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{e^{-\frac{2R\tau}{L}} - 1}{+2\cancel{R}/\cancel{L}}\left(+\frac{\cancel{R}}{\cancel{L}}\right)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{e^{-\frac{2R\tau}{L}}}{2} + \frac{1}{2} & \frac{e^{-\frac{2R\tau}{L}}}{2} - \frac{1}{2}\\ \frac{e^{-\frac{2R\tau}{L}}}{2} - \frac{1}{2} & \frac{e^{-\frac{2R\tau}{L}}}{2} + \frac{1}{2}\end{bmatrix}$$

$[1 \ 1]$

$$H e^{F\tau} x_0 = \underbrace{\begin{bmatrix} e^{-\frac{2R\tau}{L}} & e^{-\frac{2R\tau}{L}} \end{bmatrix}}_{M} x_0 = 0 \quad \forall \tau \in [0,t] \iff x_0 \in \operatorname{Ker} M = \operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$$

$\iff$ stati non osservabili in $[0,t]$

$$x_0 = \begin{bmatrix}\alpha\\ -\alpha\end{bmatrix} \quad \alpha \in \mathbb{R}$$

Stati indistinguibili

$$x(0) = x_0: \quad y(k) = HF^k x_0 + H\mathcal{R}_k u_k, \qquad k = 0, 1, \dots, t-1$$
$$x(0) = x_0': \quad y'(k) = HF^k x_0' + H\mathcal{R}_k u_k, \qquad k = 0, 1, \dots, t-1$$

Ingresso $u(0), u(1), \dots, u(t-1)$

$x(0) = x_0$: $y(k) = HF^k x_0 + HR_k u_k$, $k = 0, 1, \dots, t-1$

$x(0) = x_0'$: $y'(k) = HF^k x_0' + HR_k u_k$, $k = 0, 1, \dots, t-1$

Quando x_0' è indistinguibile da x_0?

$y(k) = y'(k) \quad \forall k = 0, 1, \dots, t-1$

$y'(k) - y(k) = 0 \quad \forall k = 0, 1, \dots, t-1$

$$\Downarrow$$

$$\begin{array}{ll} k=0: & \left\{ \begin{array}{l} H(x_0' - x_0) = 0 \\ HF(x_0' - x_0) = 0 \\ HF^2(x_0' - x_0) = 0 \\ \vdots \\ HF^{t-1}(x_0' - x_0) = 0 \end{array} \right. \end{array}$$

$k=0$, $k=1$, $k=2$, $\vdots$, $k=t-1$

$$\iff \underbrace{\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{t-1} \end{bmatrix}}_{} (x_0' - x_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\mathcal{O}_t$ = matrice di osservabilità in t passi

$$\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{t-1} \end{bmatrix} (x_0' - x_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \iff x_0' - x_0 \in \ker \mathcal{O}_t$$

Insieme di stati indistinguibili da x_0: $x_0 + \ker \mathcal{O}_t = \{x_0 + x,\ x \in \ker \mathcal{O}_t\}$

Esempi

1. $x(t+1) = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} x(t), \quad \alpha_1, \alpha_2 \in \mathbb{R}$

   $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$

2. $x(t+1) = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} x(t), \quad \alpha_1, \alpha_2 \in \mathbb{R}$

   $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$

note

1) $F = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix}$ $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$ $\alpha_1, \alpha_2 \in \mathbb{R}$

Σ è osservabile?

$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha_2 \end{bmatrix}$ rank $\mathcal{O} = 1 \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$

$\Longrightarrow \Sigma$ non è osservabile

$X_{NO} = \ker \mathcal{O} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$

2) $F = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\alpha_1, \alpha_2 \in \mathbb{R}$

Σ osservabile?

$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix}$ rank $\mathcal{O} = 2 \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$

$\Longrightarrow \Sigma$ osservabile $\forall \alpha_1, \alpha_2 \in \mathbb{R}$ (in 2 passi)

$X_{NO}(1) = \ker \mathcal{O}_1 = \ker H = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$X_{NO}(2) = \{0\}$

Esempi

1. $x(t+1) = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} x(t), \quad \alpha_1, \alpha_2 \in \mathbb{R}$

   $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$

2. $x(t+1) = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} x(t), \quad \alpha_1, \alpha_2 \in \mathbb{R}$

   $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$

note

1) $F = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \alpha_1, \alpha_2 \in \mathbb{R}$

$\Sigma = (F, H)$ non è osservabile $\forall \alpha_1, \alpha_2 \in \mathbb{R}$

$X_{NO} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Σ ricostruibile?

$\ker \mathcal{O} = X_{NO} \subseteq \ker F^2$

$$F^2 = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^2 & \alpha_1 + \alpha_2 \\ 0 & \alpha_2^2 \end{bmatrix}$$

$$\ker F^2 = \begin{cases} \{0\} & \alpha_1 \neq 0, \ \alpha_2 \neq 0 \\ \text{span}\left\{ \begin{bmatrix} 1 \\ -\alpha_1 \end{bmatrix} \right\} & \alpha_2 = 0, \ \alpha_1 \neq 0 \\ \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} & \alpha_1 = 0, \ \alpha_2 \neq 0 \\ \mathbb{R}^2 & \alpha_1 = \alpha_2 = 0 \end{cases}$$

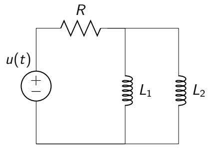
(first two cases) $X_{NO} \not\subseteq \ker F^2$, Σ non ricostruibile

(last two cases) $X_{NO} \subseteq \ker F^2$, Σ ricostruibile

2) $F = \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \alpha_1, \alpha_2 \in \mathbb{R}$

Σ osservabile $\Longrightarrow X_{NO} = \{0\} \Longrightarrow \ker F^2 \supseteq \{0\} \Longrightarrow \Sigma$ ricostruibile

Esempio



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note

$$x_1 = i_{L_1}, \quad x_2 = i_{L_2}$$

$$y = i_{L_1} + i_{L_2} = x_1 + x_2$$

$$F = \begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix} \qquad G = \begin{bmatrix} -\frac{R}{L_1} \\ -\frac{R}{L_2} \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Σ osservabile?

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -R\left(\frac{1}{L_1} + \frac{1}{L_2}\right) & -R\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \end{bmatrix}$$

$\text{rank}\, \mathcal{O} = 1 < n = 2$

$\Longrightarrow \Sigma$ non osservabile

$$X_{NO} = \ker \mathcal{O} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$