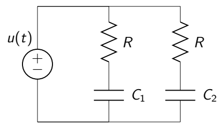
Esempio introduttivo



$x_1(t) = v_{C_1}(t)$, $x_2(t) = v_{C_2}(t)$

Se $C_1 = C_2$ e $x_1(0) = x_2(0) = 0$:

$\Rightarrow x_1(t) = x_2(t)$, $\forall u(t), \forall t \geq 0$

$\Rightarrow X_R(t) = \{x_1 = x_2\}$, $\forall t \geq 0$

note

$$x_1(t) = v_{C_1}(t) \quad , \quad x_2(t) = v_{C_2}(t)$$

$$C_1 = C_2 = C \qquad x_1(0) = x_2(0) = 0$$

Spazio raggiungibile $X_R(t)$?

$$\dot{x}_1 = \dot{v}_{C_1} = \frac{1}{C} i_{C_1} = \frac{1}{C} i_R = \frac{1}{C} \frac{u - v_{C_1}}{R} = \frac{1}{RC} u - \frac{x_1}{RC}$$

$$\dot{x}_2 = \dot{v}_{C_2} = \frac{1}{C} i_{C_2} = \frac{1}{C} \frac{u - v_{C_2}}{R} = \frac{1}{RC} u - \frac{1}{RC} x_2$$

$$x = \underbrace{\begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}}_{F} x + \underbrace{\begin{bmatrix} \frac{1}{RC} \\ \frac{1}{RC} \end{bmatrix}}_{G} u$$

$$x(t) = \cancelto{0}{e^{Ft} x_0} + \int_0^t e^{F(t-\tau)} G u(\tau)\, d\tau$$

$$= \int_0^t \frac{1}{RC} \begin{bmatrix} e^{-\frac{1}{RC}(t-\tau)} & 0 \\ 0 & e^{-\frac{1}{RC}(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(\tau)\, d\tau = \begin{bmatrix} \frac{1}{RC} \int_0^t e^{-\frac{1}{RC}(t-\tau)} u(\tau)\, d\tau \\ \frac{1}{RC} \int_0^t e^{-\frac{1}{RC}(t-\tau)} u(\tau)\, d\tau \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x_1(t) = x_2(t) \quad \forall u(t), \; t \geq 0$$

$$\Longrightarrow X_R(t) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \; x_1 = x_2 \right\} \neq \mathbb{R}^2$$

Spazio raggiungibile

$$X_R(t) = \text{spazio raggiungibile in } t \text{ passi} = \mathrm{im}(\mathcal{R}_t)$$

Teorema: Gli spazi raggiungibili soddisfano:

$$X_R(1) \subseteq X_R(2) \subseteq X_R(3) \subseteq \cdots$$

Inoltre, esiste un primo intero $i \leq n$ tale che

$$X_R(i) = X_R(j), \quad \forall j \geq i.$$

$$x(t+1) = F\,x(t) + G\,u(t)$$

$$x(t) = R_t\, u_t, \qquad X_R(t) = \mathrm{im}(R_t)$$

$$R_t = [G \;\; FG \;\; \cdots \;\; F^{t-1}G]$$

1) $\underline{X_R(t) \subseteq X_R(t+1)}$

- approccio "algebrico": $G = [g_1 \; g_2 \; \cdots \; g_m] \in \mathbb{R}^{n \times m}$

$$X_R(t) = \mathrm{im}(R_t) = \mathrm{im}\,[G \;\; FG \;\; \cdots \;\; F^{t-1}G] = \mathrm{span}\{g_1, g_2, \dots, g_m, Fg_1, \dots, Fg_m, \dots, F^{t-1}g_1, \dots, F^{t-1}g_m\}$$

span → spazio generato da tutte le possibili combinazioni lineari di vettori

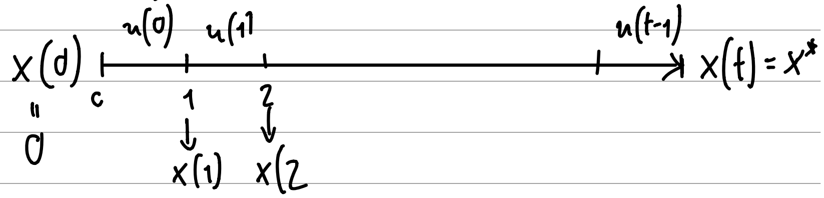
$$X_R(t+1) = \mathrm{im}(R_{t+1}) = \mathrm{im}\,[G \;\; FG \;\; \cdots \;\; F^{t}G] = \mathrm{span}\{g_1, g_2, \dots, g_m, \dots, F^{t-1}g_1, \dots, F^{t-1}g_m, F^t g_1, \dots, F^t g_m\}$$

$$\Rightarrow X_R(t) \subseteq X_R(t+1)$$

- approccio "sistemistico"

x^* raggiungibile in t passi (da $x(0) = 0$): $x^* \in X_R(t)$

$\exists\, u(0), u(1), \dots, u(t-1)$ t.c.



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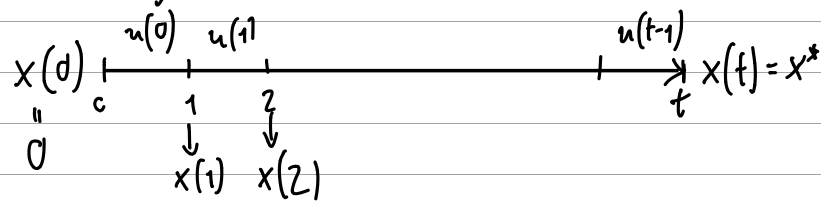
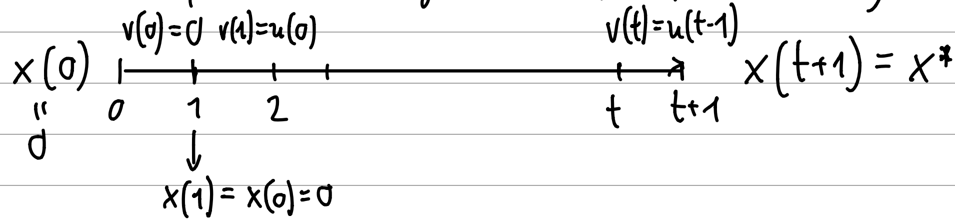


Definiamo la sequenza di ingresso: $v(0)=0$, $v(1)=u(0)$, $v(2)=u(1), \dots, v(t)=u(t-1)$



$$x^* \in X_R(t) \implies x^* \in X_R(t+1) \implies X_R(t) \subseteq X_R(t+1)$$

Esempi

1. $x(t+1) = \begin{bmatrix} \alpha_1 & 0 \\ 1 & \alpha_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$, $\alpha_1, \alpha_2 \in \mathbb{R}$

2. $x(t+1) = \begin{bmatrix} \alpha_1 & 0 \\ 1 & \alpha_2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$, $\alpha_1, \alpha_2 \in \mathbb{R}$

3. $x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$

note

Sistemi raggiungibili?

1) $F = \begin{bmatrix} \alpha_1 & 0 \\ 1 & \alpha_2 \end{bmatrix}$, $\alpha_1, \alpha_2 \in \mathbb{R}$ $\quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$R = R_2 = [G \quad FG] = \begin{bmatrix} 0 & 0 \\ 1 & \alpha_2 \end{bmatrix} \implies \operatorname{rank} R = 1 < 2 \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

$\implies \Sigma$ non è ragg. $\forall \alpha_1, \alpha_2 \in \mathbb{R}$

2) $F = \begin{bmatrix} \alpha_1 & 0 \\ 1 & \alpha_2 \end{bmatrix}$, $\alpha_1, \alpha_2 \in \mathbb{R}$ $\quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$R = [G \quad FG] = \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \end{bmatrix} \implies \operatorname{rank} R = 2 \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

$\implies \Sigma$ è raggiungibile $\forall \alpha_1, \alpha_2 \in \mathbb{R}$

3) $F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$X_R(1) = \operatorname{im} R_1 = \operatorname{im} G = \operatorname{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \mathbb{R}^3$$

$$X_R(2) = \operatorname{im} R_2 = \operatorname{im} [G \quad FG] = \operatorname{im} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \mathbb{R}^3$$

$\Downarrow$

Σ è raggiungibile in 2 passi

Raggiungibilità ed equivalenza algebrica

$$x(t+1) = Fx(t) + Gu(t) \xrightarrow{z=T^{-1}x} z(t+1) = F'z(t) + G'u(t)$$

$$F' = T^{-1}FT, \; G' = T^{-1}G$$

$$x(t+1) = Fx(t) + Gu(t) \qquad \Sigma$$

$$\downarrow z = T^{-1}x$$

$$z(t+1) = \underbrace{T^{-1}FT}_{F'} z(t) + \underbrace{T^{-1}G}_{G'} u(t) \qquad \Sigma'$$

Cosa succede alla raggiungibilità di Σ'?

$$R' = [G' \;\; F'G' \; \cdots \; (F')^{n-1}G']$$

$$= [T^{-1}G \;\; T^{-1}F\cancel{T}\cancel{T^{-1}}G \; \cdots \; T^{-1}F^{n-1}\cancel{T}\cancel{T^{-1}}G]$$

$$= [T^{-1}G \;\; T^{-1}FG \; \cdots \; T^{-1}F^{n-1}G] = T^{-1}R \qquad (*)$$

T è invertibile $\Longrightarrow \text{rank}(R') = \text{rank}(R)$

$$\Sigma \text{ ragg.} \iff \Sigma' \text{ ragg.}$$

Σ raggiungibile $\Longrightarrow \text{rank}(R) = n \Longrightarrow \det(RR^T) \neq 0$

$\Longrightarrow RR^T$ invertibile

$(*)$ $R' = T^{-1}R \longrightarrow R'R^T = T^{-1}(RR^T)$

$\longrightarrow T^{-1} = R'R^T(RR^T)^{-1}$

$\longrightarrow T = (RR^T)(R'R^T)^{-1}$

Calcolo dell'ingresso di controllo (a minima energia)

Se Σ è raggiungibile in t passi, come costruire una sequenza di ingresso $u_t \in \mathbb{R}^{mt}$ per raggiungere un qualsiasi stato $x^* \in \mathbb{R}^n$ in t passi?

$$x(t+1) = F x(t) + G u(t), \quad x(0) = x_0 \quad \Sigma$$

Σ raggiungibile in t passi

$x^* = x(t) \in \mathbb{R}^n$, come è fatto l'ingresso?

1) Caso $x_0 = 0$

$\exists\, u_t \in \mathbb{R}^{mt}$ t.c. $x^* = x(t) = R_t u_t$

Introduciamo una variabile ausiliaria $\eta_t \in \mathbb{R}^n$: $u_t = R_t^T \eta_t$

$$x^* = x(t) = R_t R_t^T \eta_t \xRightarrow[R_t R_t^T \text{ invertibile}]{} \eta_t = (R_t R_t^T)^{-1} x^*$$

$$\Longrightarrow \boxed{u_t = R_t^T \eta_t = R_t^T (R_t R_t^T)^{-1} x^*}$$

u_t è unico? In generale no

$u_t' = u_t + \bar{u}, \quad \bar{u} \in \ker(R_t)$

$$R_t u_t' = R_t(u_t + \bar{u}) = R_t u_t + \cancelto{0}{R_t \bar{u}} = x(t) = x^*$$

2) Caso $x_0 \neq 0$

$$x^* = x(t) = F^t x_0 + R_t u_t \Longrightarrow \underbrace{(x^* - F^t x_0)}_{\tilde{x}} = R_t u_t$$

$$\Longrightarrow u_t = R_t^T (R_t R_t^T)^{-1} \tilde{x}$$

$$\overset{!}{=} R_t^T (R_t R_t^T)^{-1} (x^* - F^t x_0)$$

Esempio

1. $x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$

ingressi $u'(t)$ per raggiungere $x^* = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ da $x_0 = 0$ in 2 passi?

note

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$u'(t)$ t.c. $x_0 = x(0) = 0$ $\quad x(2) = x^* = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$?

$$R_2 = [\,G \;\; FG\,] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

rank $R_2 = 3 \implies \Sigma$ è raggiungibile in 2 passi

$\implies \exists\, u'(t)$

$$u_2^* = \begin{bmatrix} u^*(1) \\ \hdashline u^*(0) \end{bmatrix} = R_2^T (R_2 R_2^T)^{-1} x^*$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Bigg)^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \hdashline 1 \\ 0 \end{bmatrix} \qquad u^*(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u^*(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\operatorname{Ker} R_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$u_2' = u_2^* + \underset{\substack{\downarrow \\ \bar{u} \in \ker R_2}}{\bar{u}} = \begin{bmatrix} 1 \\ 0 \\ \hdashline 1 \\ \alpha \end{bmatrix} \quad \alpha \in \mathbb{R} \qquad u'(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u'(0) = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$