Stima & Filtraggio: Lab 4

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Today's Lab

Wiener Filtering & Applications II

(Some other) **MATLAB**® tools for Wiener filtering

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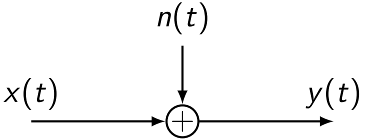
Wiener Filtering & Applications II

(Some other) **MATLAB**® tools for Wiener filtering

- Scalar rational spectral factorization
- Wiener prediction

Wiener filtering

Setup



$\{x(t)\}_{t\in\mathbb{Z}}$: stationary input process

$\{n(t)\}_{t\in\mathbb{Z}}$: stationary external noise

$\{y(t)\}_{t\in\mathbb{Z}}$: stationary output process

$$\begin{bmatrix} \Phi_x & \Phi_{xy} \\ \Phi_{xy}^* & \Phi_y \end{bmatrix}$$

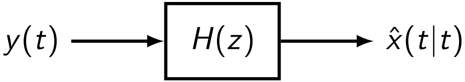
Joint spectrum

Wiener filtering

Wiener estimate

$$H(z) := \left[\frac{\Phi_{xy}(z)}{W_y(z^{-1})}\right]_+ \frac{1}{W_y(z)}$$

$$\Phi_y(e^{j\theta}) = W_y(e^{j\theta})W_y(e^{-j\theta})$$



Wiener filtering

Wiener estimate

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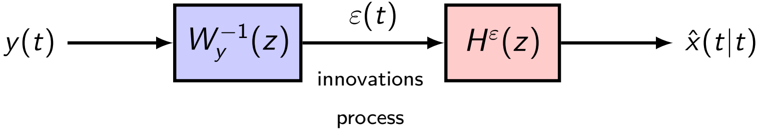


Wiener filtering

Wiener estimate

$$H(z) := \left[\frac{\Phi_{xy}(z)}{W_y(z^{-1})} \right]_+ \frac{1}{W_y(z)}$$

$$\Phi_y(e^{j\theta}) = W_y(e^{j\theta}) W_y(e^{-j\theta})$$

$$\mathrm{Var}\, \tilde{x}(t|t) = \int_{-\pi}^{\pi} \left(\Phi_x(e^{j\theta}) - H^\varepsilon(e^{j\theta}) H^\varepsilon(e^{-j\theta}) \right) \frac{\mathrm{d}\theta}{2\pi}$$

estimation error variance

Wiener filtering *in practice*

How to perform spectral factorization?

How to compute the causal and (strictly) anticausal part of a rational function?

How to compute the integral of a rational function upon the unit circle?

Wiener filtering *in practice*

How to perform spectral factorization?

How to compute the causal and (strictly) anticausal part of a rational function?

See Lab 3

How to compute the integral of a rational function on the unit circle?

See Lab 3

Scalar rational spectral factorization

$\Phi(e^{j\theta})$ spectral density of a scalar second-order stationary process $\{y(t)\}_{t\in\mathbb{Z}}$:

1. $\Phi(e^{j\theta}) = \Phi(e^{-j\theta})$
2. $\Phi(e^{j\theta}) \geq 0, \quad \forall\theta \in [-\pi, \pi]$
3. $\int_{-\pi}^{\pi} \Phi(e^{j\theta}) \frac{d\theta}{2\pi} = \mathbb{E}[y(t)^2] < \infty$

Scalar rational spectral factorization

$\Phi(e^{j\theta})$ spectral density of a scalar second-order stationary process $\{y(t)\}_{t\in\mathbb{Z}}$. Suppose that $\Phi(z)$ is a ***rational*** function of $z \in \mathbb{C}$:

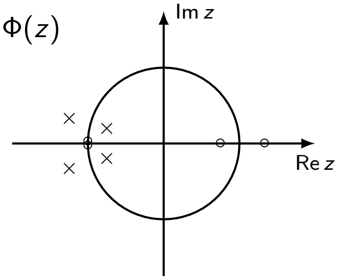
1. $\Phi(z) = \Phi(z^{-1})$ → If $\alpha \in \mathbb{C}$ is a pole (zero, resp.) of $\Phi(z)$ then $1/\bar{\alpha}$ is a pole (zero) of $\Phi(z)$ (*with the same multiplicity!*)
2. $\Phi(e^{j\theta}) \geq 0, \quad \forall \theta \in [-\pi, \pi]$ → The zeros on the unit circle of $\Phi(z)$ have even multiplicites
3. $\int_{-\pi}^{\pi} \Phi(e^{j\theta}) \frac{d\theta}{2\pi} = \mathbb{E}[y(t)^2] < \infty$ → $\Phi(z)$ has no poles on the unit circle

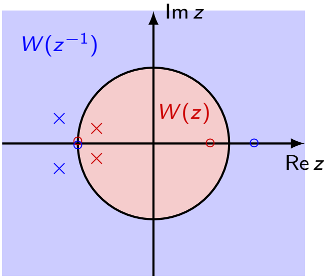
Scalar rational spectral factorization

Task. Compute a factorization of $\Phi(z)$ of the form

$$\Phi(z) = W(z)W(z^{-1}),$$

where $W(z)$ is a scalar rational function such that it is strictly stable and with marginally stable inverse. $W(z)$ is called the ***minimum-phase spectral factor*** of $\Phi(z)$.





Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^n$ be the *strictly stable* (finite) poles of $\Phi(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^m$ be the *marginally stable* (finite) zeros of $\Phi(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with *half-multiplicity* for the marginally stable zeros.

Scalar rational spectral factorization

the procedure

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1. $\Phi(z) = cz^r \dfrac{\prod_{i=1}^m (z - z_i)(z - 1/z_i)}{\prod_{i=1}^n (z - p_i)(z - 1/p_i)}$, $c > 0$, $r \in \mathbb{Z}$

(zero-pole-gain decomposition)

Scalar rational spectral factorization

the procedure

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2. $\Phi(z) = \lambda^2 \dfrac{\prod_{i=1}^m (z - z_i)(z^{-1} - z_i)}{\prod_{i=1}^n (z - p_i)(z^{-1} - p_i)}$, $\lambda^2 := c\dfrac{\prod_{i=1}^m -\frac{1}{z_i}}{\prod_{i=1}^n -\frac{1}{p_i}} > 0$

("symmetrizing" the equation)

Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^n$ be the strictly stable (finite) poles of $\Phi(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^m$ be the marginally stable (finite) zeros of $\Phi(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with half-multiplicity for the marginally stable zeros.

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3. $W(z) = z^{n-m} \lambda \dfrac{\prod_{i=1}^m (z - z_i)}{\prod_{i=1}^n (z - p_i)}$ *("balancing" relative degree)*

Scalar rational spectral factorization

...in MATLAB®

```
>> [cvZ,cvP,dK] = zpk(tfPhi,'v')
```

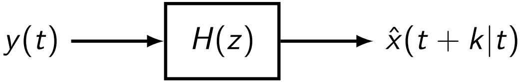
- $\texttt{cvZ} = [z_1\ z_2\ \dots\ z_{2m}]^\top$ (column vector containing the zeros of $\Phi(z)$, counted with multiplicity)
- $\texttt{cvP} = [p_1\ p_2\ \dots\ p_{2n}]^\top$ (column containing the poles of $\Phi(z)$, counted with multiplicity)
- $\texttt{dK} = c$ (gain of $\Phi(z)$)

Wiener prediction

***k*-step ahead Wiener predictor**

$$H(z) := \left[z^k \frac{\Phi_{xy}(z)}{W_y(z^{-1})} \right]_+ \frac{1}{W_y(z)}$$

$$\Phi_y(e^{j\theta}) = W_y(e^{j\theta}) W_y(e^{-j\theta})$$



Wiener prediction

k-step ahead Wiener predictor

Case $\{x(t)\}_{t\in\mathbb{Z}} = \{y(t)\}_{t\in\mathbb{Z}}$

$$H(z) := \left[z^k W_y(z)\right]_+ \frac{1}{W_y(z)}$$

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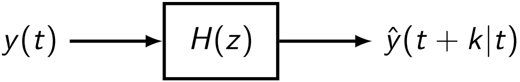

Wiener prediction

***k*-step ahead Wiener predictor**

Case $\{x(t)\}_{t\in\mathbb{Z}} = \{y(t)\}_{t\in\mathbb{Z}}$

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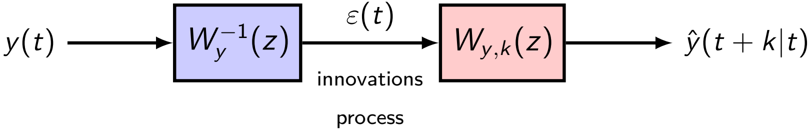

How to compute $\left[z^k W_y(z)\right]_+$?

Suppose $\Phi_y(z)$ rational:

$$W_y(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k W_y(z) = \left[z^k W_y(z)\right]_+ + \left[\left[z^k W_y(z)\right]\right]_-$$

How to compute $\left[z^k W_y(z)\right]_+$?

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$$z^k W_y(z) = \left[z^k W_y(z)\right]_+ + \left[\left[z^k W_y(z)\right]\right]_-$$

$$\left[z^k W_y(z)\right]_+ \rightarrow \frac{C_k(z)}{A(z)}$$

$$\left[\left[z^k W_y(z)\right]\right]_- \rightarrow \ell_0 z^k + \ell_1 z^{k-1} + \dots + \ell_{k-1} z$$

How to compute $\left[z^k W_y(z)\right]_+$?

Suppose $\Phi_y(z)$ rational:

$$W_y(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k \frac{C(z)}{A(z)} = \frac{C_k(z)}{A(z)} + \sum_{i=0}^{k-1} \ell_i z^{k-i}$$

How to compute $\left[z^k W_y(z)\right]_+$?

Suppose $\Phi_y(z)$ rational:

$$W_y(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k C(z) = A(z) \sum_{i=0}^{k-1} \ell_i z^{k-i} + C_k(z)$$

$z^k C(z)/zA(z) \to$ polynomial division

$Q(z) := \sum_{i=0}^{k-1} \ell_i z^{k-1-i} \to$ quotient

$R(z) := C_k(z) \to$ remainder

How to compute $\left[z^k W_y(z)\right]_+$?

Suppose $\Phi_y(z)$ rational:

$$W_y(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k C(z) = zA(z)Q(z) + R(z)$$

...in MATLAB®

```
>> [rvQ,rvR] = deconv([rvC 0 .. 0], [rvA 0])
```

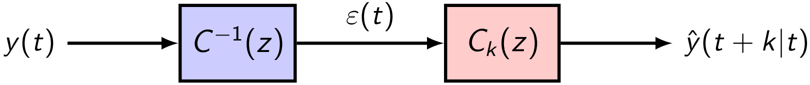
k zeros

Wiener prediction: rational case

$$H(z) := \left[z^k W_y(z)\right]_+ \frac{1}{W_y(z)} = \frac{C_k(z)}{C(z)}$$

$$\mathrm{Var}\,\tilde{y}(t+k|t) = \sum_{i=0}^{k-1} \ell_i^2$$

prediction error variance



✎ *Practice time!*

Ex 1. Create a function

```
tfW = spectralFactor(tfPhi)
```

that has as input a transfer function object `tfPhi` corresponding to a rational scalar discrete-time spectral density. The function returns the minimum-phase spectral factor of `tfPhi`.

Ex 2. Create a function

```
[cvYhat,dVar] = WienerPredictor(tfPhi,cvY,iK)
```

that has as input a rational scalar spectral density `tfPhi` of the process $\{y(t)\}_{t\in\mathbb{Z}}$, a trajectory `cvY` of the latter process, and a positive integer $\texttt{iK} = k$. The function returns the Wiener predictions $\hat{y}(t+k|t)$ in the vector `cvYhat` and the prediction error variance $\operatorname{Var}\tilde{y}(t+k|t)$ in the variable `dVar`.