Signals & Systems (3F1)

– Examples Paper 3F1/1 Discrete-Time Systems –

November 18, 2015

Standard notation

j	Imaginary unit
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{I}\mathrm{m}(a)$	Imaginary part of $a \in \mathbb{C}$
$\mathbb{R}\mathrm{e}(a)$	Real part of $a \in \mathbb{C}$
a^*	Complex conjugate of $a \in \mathbb{C}$
$\mathcal{L}\{\cdot\}$	Laplace transform
$\mathcal{Z}\{\cdot\}$	z-transform
$\delta[k]$	Kronecker's delta, for $k \in \mathbb{Z}$, $\delta[k] = 1$, if $k = 0$ $\delta[k] = 1$, otherwise
$\delta_{-1}[k]$	(Discrete time) step function, for $k \in \mathbb{Z}$, $\delta_{-1}[k] = 1$, if $k \geq 0$, $\delta_{-1}[k] = 0$ otherwise
$\delta_{-1}(t)$	Step function, for $t \in \mathbb{R}$, $\delta_{-1}(t) = 1$ if $t \geq 0$, $\delta_{-1}(k) = 0$ otherwise

Symbol legend

☞	Important fact
✍	Computations needed
🖃	Use data book
⚠	Pay attention
⯑	Clarification

Question 1. Find from first principles the z-transforms of the sequences obtained by sampling, (with uniform sampling period T), the continuous-time waveforms whose Laplace transforms are:

(i) $\dfrac{1}{s}$, (ii) $\dfrac{1}{s+a}$, (iii) $\dfrac{1}{(s+a)(s+b)}$, (iv) $\dfrac{s+a}{(s+a)^2+b^2}$.

Note that the first sample is taken at $t = 0^+$.

For each point there are essentially three steps to follow:

(i) *First step:* Inverse $\mathcal{L}$-transform

$$f(t) := \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad t \geq 0.$$

Second step: Sampling of period T

$$f(kT) = 1, \quad k \geq 0.$$

Third step: z-transform

⚠ You can omit the computation of the Region of Convergence of the z-transform, but bear in mind that the z-transform is not generally defined for all $z \in \mathbb{C}$!

$$\begin{aligned}
\mathcal{Z}\{f(kT)\} &= \sum_{k=0}^{\infty} f(kT)z^{-k} \\
&= \sum_{k=0}^{\infty} z^{-k} \\
&= \frac{1}{1-z^{-1}}, \quad |z| > 1.
\end{aligned}$$

(ii) *First step:* Inverse $\mathcal{L}$-transform

$$f(t) := \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}, \quad t \geq 0.$$

Second step: Sampling of period T

$$f(kT) = e^{-akT}, \quad k \geq 0.$$

Third step: z-transform

$$\begin{aligned}
\mathcal{Z}\{f(kT)\} &= \sum_{k=0}^{\infty} f(kT)z^{-k} \\
&= \sum_{k=0}^{\infty} e^{-akT}z^{-k} \\
&= \sum_{k=0}^{\infty} (e^{-aT}z^{-1})^k \\
&= \frac{1}{1-e^{-aT}z^{-1}}, \quad |z| > e^{-aT}.
\end{aligned}$$

(iii) *First step:* Inverse $\mathcal{L}$-transform

You can use either the partial fraction expansion or the residue method (or both for double checking!)

$$
\begin{aligned}
f(t) &:= \mathcal{L}^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{b-a}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\right\} \\
&= \frac{1}{b-a}\left(e^{-at}-e^{-bt}\right), \quad t \geq 0.
\end{aligned}
$$

Second step: Sampling of period T

$$
f(kT) = \frac{1}{b-a}\left(e^{-akT}-e^{-bkT}\right), \quad k \geq 0.
$$

Third step: z-transform

$$
\begin{aligned}
\mathcal{Z}\{f(kT)\} &= \sum_{k=0}^{\infty} f(kT) z^{-k} \\
&= \sum_{k=0}^{\infty} \frac{1}{b-a}\left(e^{-akT}-e^{-bkT}\right) z^{-k} \\
&= \frac{1}{b-a}\sum_{k=0}^{\infty}\left(e^{-aT}z^{-1}\right)^k - \frac{1}{b-a}\sum_{k=0}^{\infty}(e^{-bT}z^{-1})^k \\
&= \frac{1}{b-a}\left(\frac{1}{1-e^{-aT}z^{-1}}-\frac{1}{1-e^{-bT}z^{-1}}\right) \\
&= \frac{z^{-1}(e^{-aT}-e^{-bT})}{(b-a)(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}, \quad |z| > \max\{e^{-aT}, e^{-bT}\}.
\end{aligned}
$$

(iv) *First step:* Inverse $\mathcal{L}$-transform

You can use the Electrical and Information Data Book to save time!

$$
\begin{aligned}
f(t) &:= \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} \\
&= e^{-at}\cos(bt) \\
&= \frac{1}{2}e^{(-a+jb)t} + \frac{1}{2}e^{(-a-jb)t}, \quad t \geq 0.
\end{aligned}
$$

Second step: Sampling of period T

$$
f(kT) = \frac{1}{2}e^{(-a+jb)kT} + \frac{1}{2}e^{(-a-jb)kT}, \quad k \geq 0.
$$

Third step: z-transform

$$
\begin{aligned}
\mathcal{Z}[f(kT)] &= \sum_{k=0}^{\infty} f(kT) z^{-k} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(e^{(-a+jb)T}z^{-1}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(e^{(-a-jb)T}z^{-1}\right)^k \\
&= \frac{1}{2}\left(\frac{1}{1-e^{(-a+jb)kT}z^{-1}}+\frac{1}{1-e^{(-a-jb)kT}z^{-1}}\right) \\
&= \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}-e^{-2aT}z^{-1}}, \quad |z| > e^{-aT}.
\end{aligned}
$$

$\diamondsuit$

Question 2. A sequence is given by $u_k = 1$ for $k = 0, 1$ and zero otherwise. Find the z-transform of this sequence and hence solve the following difference equations for y_k, using the z-transform technique.

(i) $y_k = u_k + u_{k-1} + u_{k-2}$, (FIR filter).

(ii) $y_k = 0.8y_{k-1} + 0.2u_k,\ y_{-1} = 0$, (Exponential Smoother).

(iii) $y_k = 0.98y_{k-1} - 0.9604y_{k-2} + u_k,\ y_{-1} = 0,\ y_{-2} = 1$, (IIR filter).

Since $u_k = 0$ for $k \neq 0, 1$, by applying the definition of z-transform we get

$$U(z) := \mathcal{Z}\{u_k\} = \sum_{k=0}^{\infty} u_k z^{-k} = 1 + z^{-1}.$$

(i) By taking the z-transform of both sides

⚠ Recall the linearity and time-delay properties of z-transform!

$$\begin{aligned} Y(z) := \mathcal{Z}\{y_k\} &= \mathcal{Z}\{u_k\} + \mathcal{Z}\{u_{k-1}\} + \mathcal{Z}\{u_{k-2}\} \\ &= U(z) + \cancel{u_{-1}} + z^{-1}U(z) + \cancel{u_{-2}} + \cancel{u_{-1}}z^{-1} + z^{-2}U(z) \\ &= 1 + z^{-1} + z^{-1} + z^{-2} + z^{-2} + z^{-3} \\ &= 1 + 2z^{-1} + 2z^{-2} + z^{-3}. \end{aligned}$$

Hence, we obtain

$$y_k = \delta[k] + 2\delta[k-1] + 2\delta[k-2] + \delta[k-3].$$

or equivalently, in terms of sequence, $\{y_k\}_{k=0}^{\infty} = \{1, 2, 2, 1, 0, \dots\}$.

(ii) As before, we take the z-transform of both sides

$$\begin{aligned} Y(z) &= 0.8(z^{-1}Y(z) + \cancel{y_{-1}}) + 0.2U(z), \\ &= 0.8z^{-1}Y(z) + 0.2(1 + z^{-1}) \end{aligned}$$

and, by rearranging the latter equation, we arrive at

$$Y(z) = 0.2 + \frac{0.36z^{-1}}{1 - 0.8z^{-1}}.$$

Hence,

$$y_k = 0.2\delta[k] + 0.36(0.8)^{k-1}\delta_{-1}[k-1],$$

so that $\{y_k\}_{k=0}^{\infty} = \{0.2, 0.36, 0.36 \cdot 0.8, 0.36(0.8)^2, \dots, 0.36(0.8)^k, \dots\}$.

(iii) The z-transform of both sides gives

$$\begin{aligned} Y(z) &= 0.98(z^{-1}Y(z) - \cancel{y_{-1}}) - 0.9604(z^{-2}Y(z) - z^{-1}\cancel{y_{-1}} - y_{-2}) + U(z) \\ &= 0.98z^{-1}Y(z) - 0.9604(z^{-2}Y(z) - 1) + 1 + z^{-1} \end{aligned}$$

and a rearrangement of the terms yields

$$Y(z) = \frac{0.0396 + z^{-1}}{1 - 0.98z^{-1} + 0.9604z^{-2}}.$$

Now notice that the denominator of $Y(z)$ can be written as

$$1 - 0.98z^{-1} + 0.9604z^{-2} = 1 - 2 \cdot 0.98\cos(\pi/3)z^{-1} + 0.98^2z^{-2}.$$

Given an expression of the form

$$Y(z) = \frac{a + bz^{-1}}{c + dz^{-1} + ez^{-2}}$$

with a, b, c, d, $e \in \mathbb{R} \setminus \{0\}$. We can always rewrite it as

$$Y(z) = \frac{a}{c} \cdot \frac{1 + \frac{b}{a} z^{-1}}{1 + \frac{d}{c} z^{-1} + \frac{e}{c} z^{-2}}.$$

If the polynomial $c + dz^{-1} + ez^{-2}$ has complex-conjugate roots then

$$\Delta = \frac{d^2}{c^2} - 4\frac{e}{c} < 0,$$

which implies that e and c must have the same sign and $\left|\frac{d}{2\sqrt{ec}}\right| < 1$. Now, set

$$r := \sqrt{\frac{e}{c}}, \quad \omega_0 := \arccos\left(-\frac{d}{2\sqrt{ec}}\right).$$

Then we can rewrite the initial expression as

$$Y(z) = \frac{a}{c} \cdot \frac{1 + \frac{b}{a} z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}.$$

Hence we obtain

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$$\begin{aligned} y_k &= \frac{(0.98)^{k-1}}{\sin(\pi/3)} \left(0.98 \cdot 0.0396 \sin((k+1)\pi/3) + \sin(k\pi/3)\right) \delta_{-1}[k] \\ &= (0.98)^{k-1} \left(0.0448 \sin((k+1)\pi/3) + 1.1547 \sin(k\pi/3)\right) \delta_{-1}[k]. \end{aligned}$$

and we are done.

As a final check it is worth to compare the value of y_0 (and y_1, y_2,...) with the one obtained from the difference equation.

$\diamondsuit$

Question 3. For the three difference equations of question 2, write down the z-plane transfer function relating $\{y_k\}$ to $\{u_k\}$.

(a) Evaluate their poles and zeros, and calculate and sketch the response to a unit step on u_k. (Assume $y_k = 0$, $k < 0$).

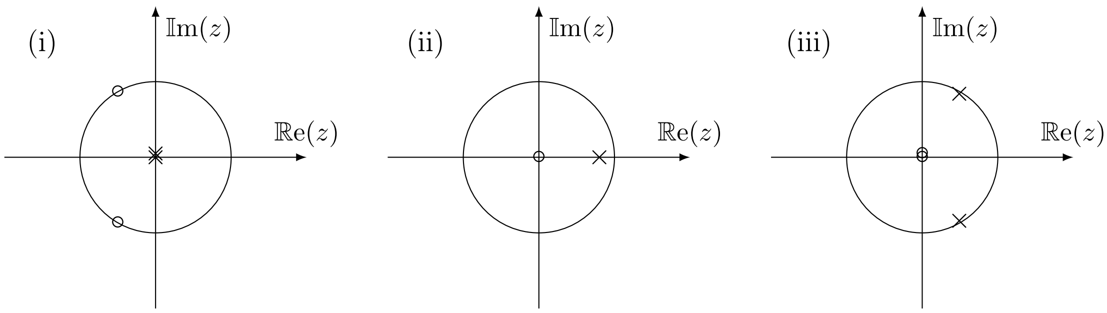
(b) Assuming the time between samples is T, calculate and sketch the steady state response to $u_k = \cos \omega kT$ for $\omega T = 0, \pi/3, \pi$.

The z-plane transfer function relating $\{y_k\}$ to $\{u_k\}$ is given by $G(z) = Y(z)/U(z)$, the zeros are given by the roots of the numerator $G(z)$ and the poles by the roots of the denominator of $G(z)$. Hence, with reference to Question 2, we have (see also the zero-pole plots below)

(i) $G_1(z) = 1 + z^{-1} + z^{-2} = \frac{z^2+z+1}{z^2}$, with zeros $\left\{-\frac{1}{2} + j\frac{\sqrt{3}}{2}, -\frac{1}{2} - j\frac{\sqrt{3}}{2}\right\}$ and poles $\{0, 0\}$.

(ii) $G_2(z) = \frac{0.2}{1-0.8z^{-1}} = \frac{0.2z}{z-0.8}$, with one zero at 0 and one pole at 0.8.

(iii) $G_3(z) = \frac{1}{1-0.98z^{-1}+0.9604z^{-2}} = \frac{z^2}{z^2 0.98z+0.9604}$, having zeros $\{0,0\}$ and poles $\{0.98e^{+j\pi/3}, 0.98e^{-j\pi/3}\}$.



(a) Recall that the z-transform of the discrete unit step is given by $U(z) = \frac{1}{1-z^{-1}}$. Hence the step responses for the three cases are:

(i)

$$\begin{aligned} Y_1(z) &= G_1(z)U(z) \\ &= (1+z^{-1}+z^{-2})\frac{1}{1-z^{-1}} \\ &= 1 - z^{-1} + \frac{3z^{-1}}{1-z^{-1}}, \end{aligned}$$

so that

$$y_1(k) = \delta[k] - \delta[k-1] + 3\delta_{-1}[k-1],$$

or in terms of sequence $\{y_1(k)\}_{k=0}^{\infty} = \{1, 2, 3, 3, 3, 3, \ldots\}$.

☞ Note that the steady value is attained after a finite number of steps (in this case 2). This phenomenon is a unique feature of discrete-time systems!

(ii)

$$\begin{aligned} Y_2(z) &= G_2(z)U(z) \\ &= \frac{1}{1-z^{-1}} \cdot \frac{0.2}{1-0.8z^{-1}} \\ &= \frac{1}{1-z^{-1}} - \frac{0.8}{1-0.8z^{-1}} \end{aligned}$$

and transforming back in the time-domain

$$y_2(k) = \delta_{-1}[k] - (0.8)^{k+1}\delta_{-1}[k].$$

(iii)

$$\begin{aligned} Y_3(z) &= G_3(z)U(z) \\ &= \frac{1}{1-0.98z^{-1}+0.9604z^{-2}} \cdot \frac{1}{1-z^{-1}} \\ &= 0.012 \cdot \frac{1+80z^{-1}}{1-2\cdot 0.98\cos(\pi/3)z^{-1}+0.98^2z^{-2}} + \frac{1.02}{1-z^{-1}} + 1 \end{aligned}$$

and therefore

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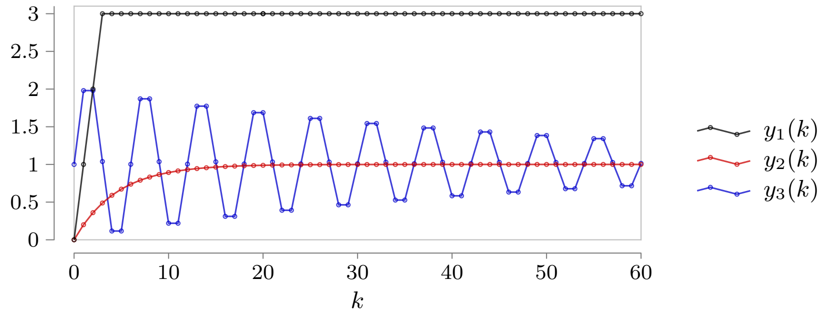
$$y_3(k) = (0.98)^{k-1}\left(0.0136\sin((k+1)\pi/3) - 0.0111\sin(k\pi/3)\right)\delta_{-1}[k] \\ + 1.02\delta_{-1}[k-1] + \delta[k].$$

The figure below shows the step responses of the three systems.



(b) The steady state response to an input $u_k = \cos(\omega kT)$ is given by

$$y_k = |G(e^{j\omega T})|\cos(\omega kT + \angle G(e^{j\omega T})).$$

Hence with reference to the previous three cases we have:

☞ Observe that $\omega = \pi/T$ corresponds to the maximum frequency for a discrete-time system!

(i)

$$y_1(k) = \begin{cases} |G(1)|\cos(\angle G(1)) = 3, & \omega T = 0, \\ |G(e^{j\pi/3})|\cos\left(\frac{\pi}{3}k + \angle G(e^{j\frac{\pi}{3}})\right) = 2\cos\left((k-1)\frac{\pi}{3}\right), & \omega T = \frac{\pi}{3}, \\ |G(-1)|\cos(\pi k + \angle G(-1)) = (-1)^k, & \omega T = \pi. \end{cases}$$

(ii)

$$y_2(k) = \begin{cases} |G(1)|\cos(\angle G(1)) = 1, & \omega T = 0, \\ |G(e^{j\pi/3})|\cos\left(\frac{\pi}{3}k + \angle G(e^{j\frac{\pi}{3}})\right) = 0.218\cos\left(k\frac{\pi}{3} - 0.857\right), & \omega T = \frac{\pi}{3}, \\ |G(-1)|\cos(\pi k + \angle G(-1)) = (-1)^k/9, & \omega T = \pi. \end{cases}$$

(iii)

☞ Notice that $\omega = \pi/(3T)$ corresponds to the resonant frequency of the system!

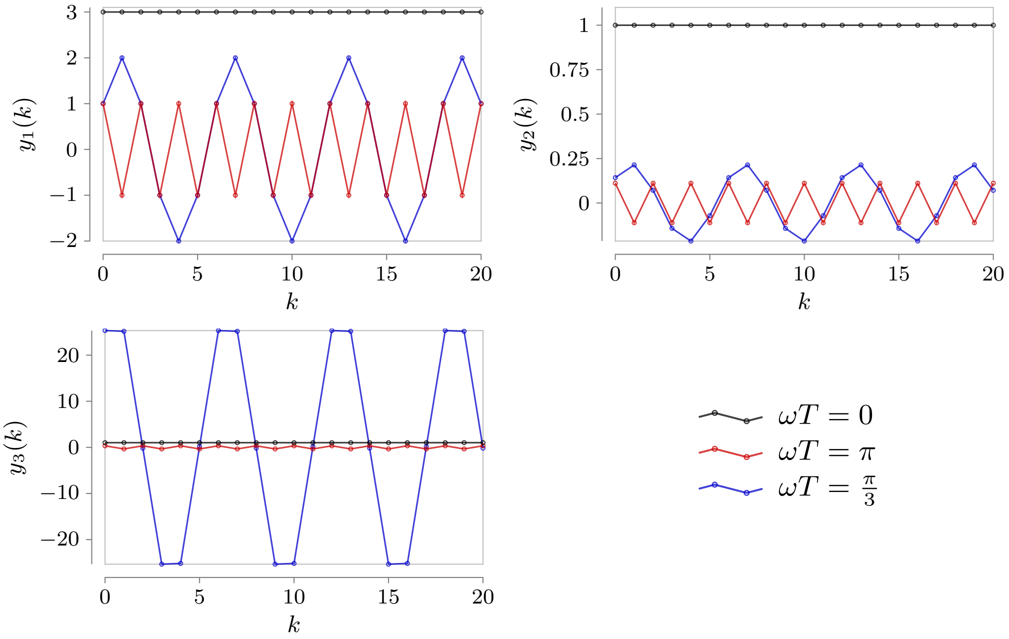
$$y_3(k) = \begin{cases} |G(1)|\cos(\angle G(1)) = 1.02, & \omega T = 0, \\ |G(e^{j\pi/3})|\cos\left(\frac{\pi}{3}k + \angle G(e^{j\frac{\pi}{3}})\right) = 29.16\cos\left(k\frac{\pi}{3} - 0.518\right), & \omega T = \frac{\pi}{3}, \\ |G(-1)|\cos(\pi k + \angle G(-1)) = 0.34\cdot(-1)^k, & \omega T = \pi. \end{cases}$$

The figure below shows the steady state responses of the three systems for $\omega T = 0, \pi/3, \pi$.

Since

$$e^{j\omega T} = e^{j\omega T + 2\pi k}, \quad \forall k \in \mathbb{Z},$$

$G(e^{j\omega T})$ is a periodic function of ω of period $T/2\pi$. Hence we can study the frequency behaviour (magnitude and phase) of $G(z)$ in the frequency interval $[-\pi/T, \pi/T]$. Furthermore, it holds $G(e^{j\omega T}) = G^*(e^{-j\omega T})$, so that:



1. the magnitude of G is an even function of the frequency, i.e.

$$|G(e^{j\omega T})| = |G(e^{-j\omega T})|,$$

2. the phase of G is an odd function of the phase, i.e.

$$\angle G(e^{j\omega T}) = -\angle G(e^{-j\omega T}).$$

Hence, due to these symmetries, the study of the frequency behaviour of $G(z)$ can be restricted to the frequency interval $[0, \pi/T]$, which implies that π/T corresponds to the maximum frequency for a discrete-time system.

$\diamondsuit$

Question 4. *(Supply and demand cycles).* In agriculture a one season prediction of price is required by the farmer to determine how much product to produce. The price at harvesting will then depend on the supply via the demand curve. Assume that the demand at time k is given by

$$d_k = d_e - ap_k$$

where p_k is the price at time k. Now assume that the supply is given by

$$s_k = s_e + b\hat{p}_k,$$

where $\hat{p}_k$ is the predicted price at time k. (d_e and s_e are constants). The price then adjusts to equate supply and demand at time k (i.e. $s_k = d_k$). Let $c = b/a$ and determine conditions on c for the stability of this system (i.e. the stability of the difference equation determining p_k), for

(i) $\hat{p}_k = p_{k-1}$, and

(ii) $\hat{p}_k = 2p_{k-1} - p_{k-2}$ (i.e. a linear extrapolation through the last two prices).

In the following we assume that a, $b \in \mathbb{R}$ are positive constants.

(i) In this case the supply takes the form

$$s_k = s_e + bp_{k-1},$$

hence, by equating supply and demand, we get

$$d_e - ap_k = s_e + bp_{k-1} \Longrightarrow p_k - \frac{b}{a}p_{k-1} = \frac{d_0 - s_0}{a}.$$

The latter difference equation is stable if the "characteristic polynomial" $p(z) := z - b/a$ has roots (strictly) inside the unit circle. Thus the system is stable if $c := b/a < 1$.

☞ Here we use the term stable to mean that the response of a dynamic system to any finite *initial conditions* is bounded.

The solution of a difference equation

$$\sum_{i=0}^{n} a_i y(k-i) = \sum_{j=0}^{m} b_j u(k-j), \quad y(-n) = y_{-n}, \dots, y(-1) = y_{-1}$$

can always be decoupled as

$$y(k) = y_\ell(k) + y_f(k),$$

where y_ℓ is the "free evolution" of the system, i.e. the evolution of the system without any inputs which depending only on the initial conditions, and y_f is the "forced evolution" of the system, i.e. the evolution of the system due to the selected input $\{u_k\}$. In particular, the free evolution has the form

$$y_\ell(k) = c_1(z_1)^k + c_2(z_2)^k + \cdots + c_n(z_n)^k,$$

where $z_1, \dots, z_n$ are the roots of the characteristic polynomial

$$p(z) = \sum_{i=0}^{n} a_i z^{n-i}$$

and $c_1, \dots, c_n \in \mathbb{C}$ are constant coefficients depending on the initial conditions.

(ii) In this case the supply takes the form

$$s_k = s_e + b(2p_{k-1} - p_{k-2}).$$

Equating supply and demand gives

$$d_e - ap_k = s_e + s_e + b(2p_{k-1} - p_{k-2}) \Longrightarrow p_k + 2cp_{k-1} - cp_{k-2} = \frac{d_0 - s_0}{a}.$$

In this case the characteristic polynomial is $p(z) := z^2 + 2cz - c$. The roots of $p(z)$ are given by $z_{1,2} = -c \pm \sqrt{c^2 + c}$. Notice that, since for $c = 0$ the two roots collapse at the point $z = 0$, the system is stable for c sufficiently small. As c increases, the negative root first crosses the unit circle at $z = -1$. Hence, solving for c the equation $-c - \sqrt{c^2 + c} = -1$ gives $c = 1/3$. We conclude that the system is stable if $c < 1/3$.

☞ Observe that this price extrapolation tends to destabilise the system!

$\diamondsuit$

Question 5. *(Numerical solutions of differential equations).*

(a) Euler's method for solving the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t)) \tag{1}$$

is to make the approximation,

$$x((k+1)T) \approx x(kT) + Tf(x(kT)) \tag{2}$$

for $k = 0, 1, 2, \ldots$, where T is the step length. Assuming $f(x) = ax$ for $a < 0$, what range of values of T in (2) will ensure that $x(kT) \to 0$ as $k \to \infty$?

(b) Euler's method is inaccurate unless T is very small and an alternative is to consider higher-order extrapolation based on $x(kT)$ and $x((k-1)T)$ and $\dot{x}(kT) = f(x(kT))$. The function

$$g(t) = (c - b + mT)\frac{t^2}{T^2} + mt + b$$

is the quadratic function that satisfies

$$g(0) = b,\ \dot{g}(0) = m,\ g(-T) = c.$$

Now if we let

$$b = x(kT),\ m = f(x(kT)),\ c = x((k-1)T)$$

then $x((k+1)T) = g(T) = x((k-1)T) + 2Tf(x(kT))$ is an extrapolation of the next value of x based on this quadratic approximation. If $f(x) = ax$ with $a < 0$ show that the method would be unstable for any $T > 0$ (!). What would be the nature of the instability?

(a) Assuming $f(x) = ax$ the difference equation becomes

$$\begin{aligned} &x((k+1)T) = x(kT) + Tax(kT) = (1 + aT)x(kT) \\ \Rightarrow\ &x((k+1)T) - (1 + aT)x(kT) = 0. \end{aligned}$$

The characteristic polynomial of the difference equation is $p(z) := z - 1 - aT$. To ensure stability the roots of $p(z)$ must be (strictly) inside the unit circle. From this fact, we obtain the condition $|1 + aT| < 1$ which in turn implies the system is stable if

$$0 < T < -\frac{2}{a}, \quad a < 0.$$

(b) For $f(x) = ax$, the modified equation takes the form

$$\begin{aligned} &x((k+1)T) = g(T) = 2Tax(kT) + x((k-1)T) \\ \Rightarrow\ &x((k+1)T) - 2Tax(kT) - x((k-1)T) = 0. \end{aligned}$$

In this case, the roots of the characteristic polynomial $p(z) := z^2 - 2Tz - 1$ are given by $z_{1,2} = aT \pm \sqrt{1 + a^2T^2}$ and for all $aT < 0$ the root z_2 is always less than -1. Hence the system is always unstable for $a < 0$. We conclude that this type of extrapolation destabilises the system, since the approximation error grows exponentially as $(z_2)^k$.

$\diamondsuit$

Question 6. A discrete time system has impulse response $\{g_k\}$ and transfer function $G(z)$.

(a) Show that $\sum_{k=0}^{\infty} |g_k|$ is finite if all the poles of $G(z)$ lie strictly inside the unit circle.

(b) Suppose $\sum_{k=0}^{\infty} |g_k|$ is infinite. Explain how the output can be made arbitrarily large at some time instant, using an input of plus and minus ones.

(c) Let $G(z) = 1/(z^2 - \sqrt{2}z + 1)$. Find a bounded input which gives an unbounded output.

(a) $G(z)$ can be written in the factorised form

⚠ $G(z)$ is supposed to be a general rational function, either improper ($m > n$) or proper ($m \leq n$).

$$G(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{(1 - p_1 z^{-1})^{n_1}(1 - p_2 z^{-1})^{n_2} \cdots (1 - p_r z^{-1})^{n_r}},$$

where p_i, $i = 1, \ldots, r$ are the stable poles of $G(z)$, i.e. $|p_i| < 1$ for all $i = 1, \ldots, r$. Furthermore, $G(z)$ can be split into partial fractions of the form

$$G(z) = \sum_{i=1}^{r} \sum_{\ell=1}^{n_r} \frac{\alpha_{i\ell}}{(1 - p_i z^{-1})^{\ell}} + \sum_{i=0}^{m-n} \beta_i z^i,$$

with $n := n_1 + n_2 + \cdots + n_r$. Now since

$$\mathcal{Z}^{-1}\left\{\frac{1}{(1 - p_i z^{-1})^{\ell}}\right\} = \frac{(k + \ell + 1)!\, p_i^k}{k!(\ell - 1)!}\delta_{-1}[k], \quad \mathcal{Z}^{-1}\{z^i\} = \delta[k + i],$$

and the series

$$\sum_{k=0}^{\infty} \frac{(k + \ell + 1)!\, p_i^k}{k!(\ell - 1)!}|p_i|^k = \frac{1}{(1 - |p_i|)^{\ell}} < \infty$$

converges to a finite value due to the fact that $|p_i| < 1$ for all $i = 1, \ldots, r$, then we have

$$\sum_{k=0}^{\infty} |g_k| \leq \sum_{i=1}^{r} \sum_{\ell=1}^{n_i} \frac{|\alpha_{i\ell}|}{(1 - |p_i|)^{\ell}} + |\beta_0| < \infty.$$

(b) Since $\sum_{k=0}^{\infty} |g_k|$ is infinite then, for all $M > 0$ there exists $N > 0$ such that

$$\sum_{k=0}^{N} |g_k| > M$$

Now the output at time N of the system described by the pulse response $\{g_i\}_{i=0}^{\infty}$ is given by the convolution sum

$$y_N = \sum_{k=0}^{N} g_k u_{N-k}.$$

Therefore, if we set $u_{N-k} := \mathrm{sign}(g_k)$, $k = 0, 1, \ldots, N$ we have

$$y_N = \sum_{k=0}^{N} |g_k| > M.$$

(c) We observe that $G(z)$ has two complex conjugate poles on the unit circle, precisely $p_{1,2} := e^{\pm j\pi/4}$. Now we choose a bounded input which in the z-transform domain has a pair of poles at the same locations $p_{1,2}$, e.g.

☞ The location of the poles identifies the resonant frequency of the system!

$$u_k = \sin(k\pi/4)\delta_{-1}[k].$$

In the z-domain, we have

$$G(z)U(z) = \frac{z^{-3}}{\sqrt{2}(1 - e^{+j\pi/4}z^{-1})^2(1 - e^{-j\pi/4}z^{-1})^2}$$

and, using the partial fractions decomposition, we arrive at

$$Y(z) = G(z)U(z) = \sum_{\ell=1}^{2} \frac{\alpha_{1,\ell}}{(1 - e^{+j\pi/4}z^{-1})^\ell} + \sum_{\ell=1}^{2} \frac{\alpha_{1,\ell}^*}{(1 - e^{-j\pi/4}z^{-1})^\ell}.$$

Eventually, by recalling the inverse formula

$$\mathcal{Z}^{-1}\left\{\frac{1}{(1 - p_i z^{-1})^\ell}\right\} = \frac{(k+\ell+1)!\, p_i^k}{k!(\ell-1)!}\delta_{-1}[k],$$

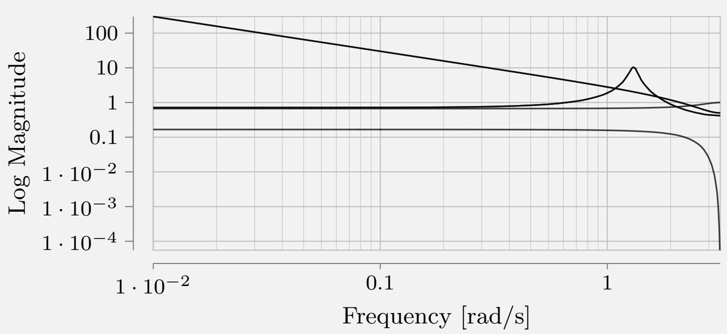
we conclude that the output y_k will include terms of the form $\frac{(k+\ell+1)!\, e^{\pm jk\pi/4}}{k!(\ell-1)!}\delta_{-1}[k]$ which real parts are unbounded in k.

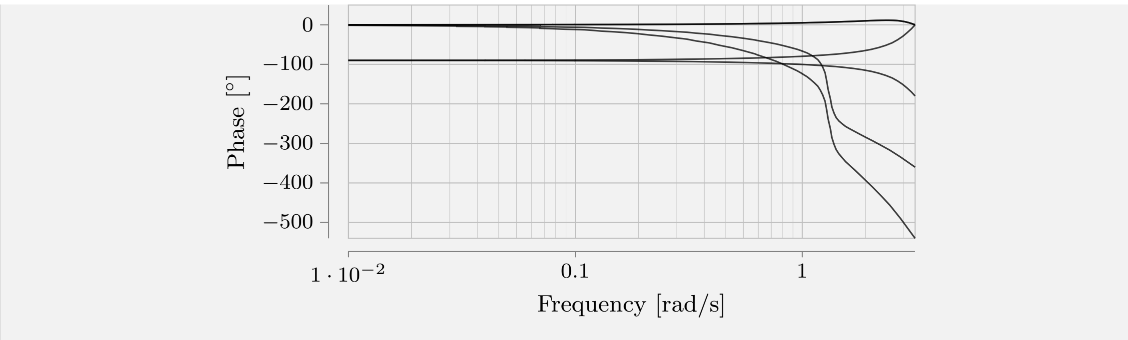
♢

Question 7. For a sampling period of 1 sec the Bode plots of the following transfer functions were plotted (see figure below). As usual, however, it was forgotten to label the graphs. Can you help?

$$(1)\ \frac{z+2}{z-1}, \qquad (2)\ \frac{2z+1}{z-1}, \qquad (3)\ \frac{1}{z^2 - 0.5z + 0.9},$$

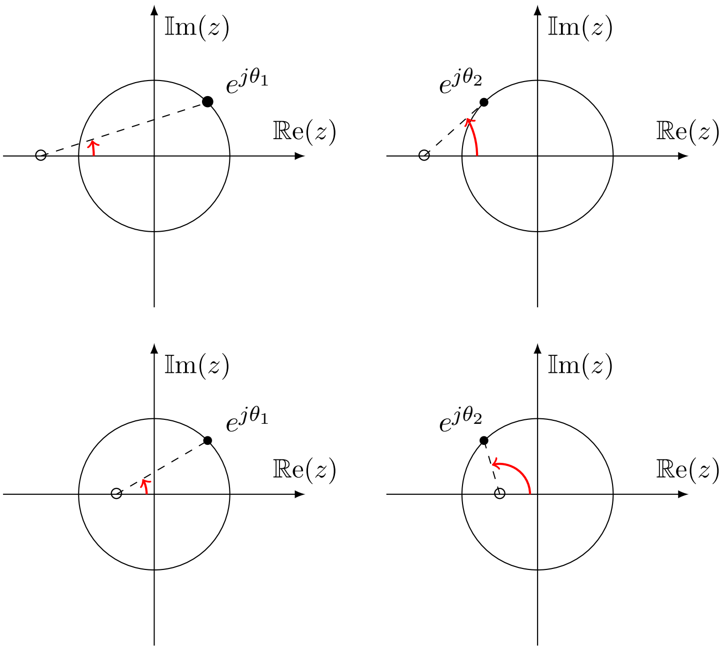
$$(4)\ \frac{1}{z(z^2 - 0.5z + 0.9)}, \qquad (5)\ \frac{(z+1)^2}{4(z+3)(z+0.5)}, \qquad (6)\ \frac{3z+1}{4z+2}.$$



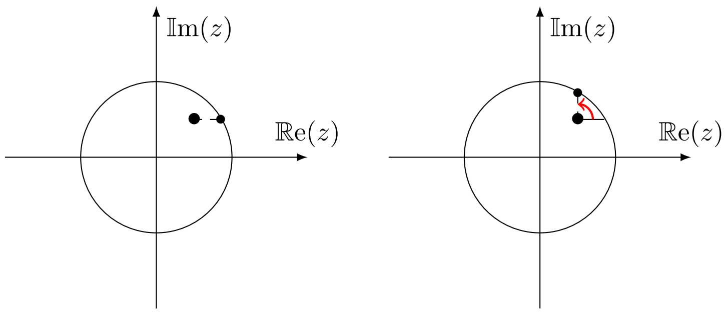


A trivial way to figure out the correct matchings is to compute the magnitude and phase of the various transfer functions at some (meaningful) frequencies (e.g. around 1 rad/s). Nevertheless, we can arrive at the same conclusions using a smarter and quicker way, as described below.

First, notice that (1) and (2) have both infinite gain at frequency 0 ($z = e^{j0} = 1$). Since there is only one gain plot that exhibits such characteristic, it follows that they must have the same gain at all frequency (indeed it holds $|e^{j\theta} + 2| = |2e^{j\theta} + 1|$). Moreover the phase plots of (1) and (2) start at $-90°$ (since they both have one pole at $z = 1$). At the maximum frequency π the phase of (1) is $-180°$ (the zero outside the unit circle gives no net change of phase from 0 rad/s to π rad/s), while the phase of (1) is $0°$ (due to the zero inside the unit circle that adds a phase of $+\pi/2$ rad).



Transfer functions (3) and (4) have the same magnitudes since they differ only by the factor z (indeed $|z| = 1$ for $e^{j\theta}$). Both have poles at $z = re^{j\theta}$ with $r^2 = 0.9$, and so $r = 0.95$ is very close to the unit circle. Hence the gain will have a peak near frequency ϕ. Both phases decrease quickly near to the resonant frequency ϕ (indeed the vector $e^{j\theta} - e^{j\phi}$ flips by nearly of 180°). Finally the factor z in the denominator of (4) gives an extra phase of $-180°$ at $\theta = \pi$.

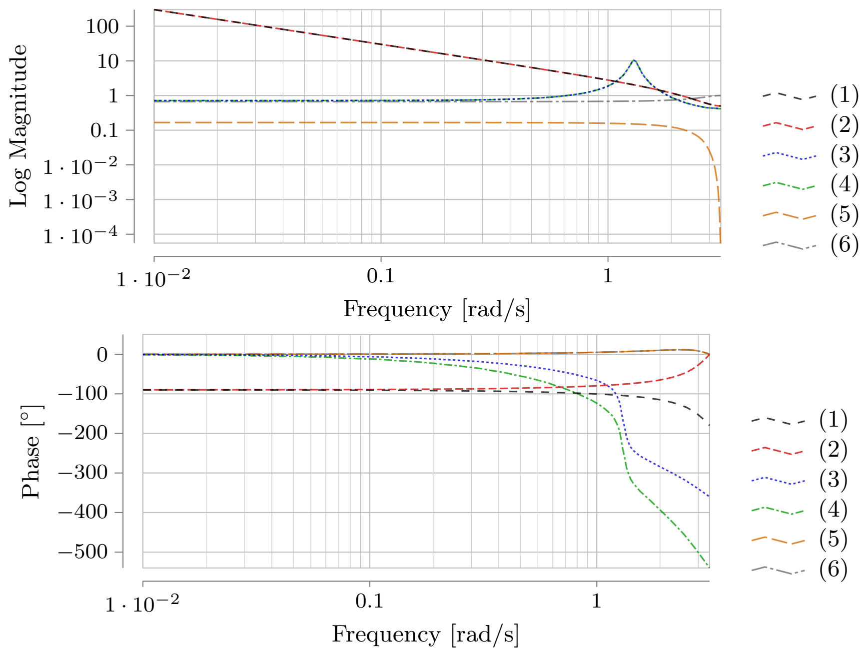


The quotient of (5) and (6) evaluated at $z = e^{j\theta}$ is positive real, indeed

$$\frac{(e^{j\theta}+1)^2}{(e^{j\theta}+3)(3e^{j\theta}+1)} = \frac{\cos\theta+1}{3\cos\theta+5}.$$

Hence the phase plots of (5) and (6) must coincide, and so it can be easily identified. Moreover $G_5(-1) = 0$ so the gain of (5) rolls down to $-\infty$ on the log scale at high frequencies. This allows to identify the gain plots of (5) and (6).

☞ You can use the MATLAB command dbode or ltiview to check your results!



◇

Question 8. Find the forward difference, backward difference and Tustin transformation of the analog low-pass filter

$$G(s) = \frac{a}{s+a}$$

($a > 0$) assuming a sampling period of T seconds. What conditions must aT satisfy for these digital filters to be stable? For what range of values of aT would these filters be reasonable approximations of $G(s)$?

(i) In the forward difference method we have $s = \frac{z-1}{T}$, hence

$$G(z) = \frac{a}{\frac{z-1}{T} + a} = \frac{aT}{z - 1 + aT}.$$

To ensure stability it must be $|1 - aT| < 1$, so that $0 < aT < 2$.

(ii) In the backward difference method we have $s = \frac{z-1}{zT}$, hence

$$G(z) = \frac{a}{\frac{z-1}{zT} + a} = \frac{aTz}{(1 + aT)z - 1}.$$

The filter is stable for all aT.

(iii) The Tustin transformation is defined as $s = \frac{2}{T}\frac{z-1}{z+1}$, hence

$$G(z) = \frac{a}{\frac{2}{T}\frac{z-1}{z+1} + a} = \frac{aT(z + 1)}{z(2 + aT) + aT - 2}.$$

The filter has a pole at $z = \frac{2-aT}{2+aT}$ and so the filter is stable for all aT.

Finally we observe that the digital filters make good sense approximations of the analog low-pass filter if the sampling frequency $1/T$ Hz is at least 2 times greater than the 3dB cutoff frequency of the filter, which for a low-pass filter is given by $f_c := a/2\pi$ Hz. This is a consequence of the Nyquist-Shannon sampling theorem which states that for a band-limited signal of band B the sample period for its perfect reconstruction must satisfy $1/T \geq 2B$. Hence, since the band of the filter can be considered approximately equal to f_c, we conclude that if

$$\frac{1}{T} \gg 2f_c \Rightarrow aT \ll \pi,$$

then the digital filters can be considered good approximations of the analog one.

☞ Note that the backward difference and Tustin transformation give stable filters while the forward difference method can return an unstable filter.

✍ The 3dB cutoff frequency of a filter $G(s)$ is computed by equating $|G(j\omega)| = \frac{1}{\sqrt{2}}$.

♢

Question 9. A motor driving a rotating inertia has transfer function $1/s(s + 1)$ from the motor current input to the position output. The output is sampled with period T, and the current input is held constant between sampling points. Show that the equivalent discrete-time system, from the sequence of current inputs to the sampled outputs, has the z-plane transfer function

$$G(z) = \frac{\left(e^{-T} - 1 + T\right) z^{-1} + \left(1 - (1 + T)e^{-T}\right) z^{-2}}{\left(1 - z^{-1}\right)\left(1 - e^{-T} z^{-1}\right)}.$$

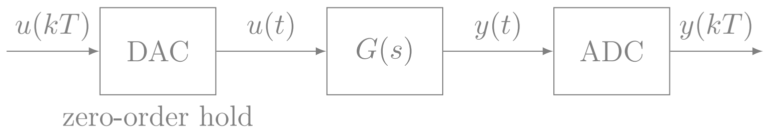
The equivalent discrete-time model can be evaluated using the step response equivalence:

$$\begin{aligned}
Y(z) &= \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{1}{s^2(s+1)} \right\}\Bigg|_{t=kT} \right\} \\
&= \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s} + \frac{1}{s+1} \right\}\Bigg|_{t=kT} \right\} \\
&= \mathcal{Z}\left\{ \left(t - 1 + e^{-t}\right)\Big|_{t=kT} \right\} \\
&= \mathcal{Z}\left\{ kT - 1 + e^{-kT} \right\} \\
&= \frac{Tz^{-1}}{(1 - z^{-1})^2} - \frac{1}{(1 - z^{-1})} + \frac{1}{(1 - e^{-T} z^{-1})}.
\end{aligned}$$

Therefore the transfer function of the equivalent discrete-time system is given by

$$
\begin{aligned}
G(z) &= (1 - z^{-1})Y(z) \\
&= \frac{Tz^{-1}}{(1 - z^{-1})} - 1 + \frac{1 - z^{-1}}{(1 - e^{-T}z^{-1})} \\
&= \frac{\left(e^{-T} - 1 + T\right) z^{-1} + \left(1 - (1+T)e^{-T}\right) z^{-2}}{(1 - z^{-1})\,(1 - e^{-T}z^{-1})}.
\end{aligned}
$$

Consider the following block diagram



First consider the zero-order hold (for short ZOH). This takes the discrete value $u(kT)$ at sample time kT and produces a rectangular pulse of height $u(kT)$ and width T. For each sample $u(kT)$, the output of the ZOH is therefore a step of height $u(kT)$ at time kT plus a step of height $-u(kT)$ at time $(k+1)T$, so the continuous signal at the output of the ZOH has the Laplace transform:

$$
\begin{aligned}
\mathcal{L}\{u(kT)\delta_{-1}(t - kT) - u(kT)\delta_{-1}(t - (k+1)T)\} &= e^{-kTs}\frac{u(kT)}{s} - e^{-(k+1)Ts}\frac{u(kT)}{s} \\
&= u(kT)\frac{1 - e^{-Ts}}{s}e^{-kTs}.
\end{aligned}
$$

So the continuous part of the system is driven by a signal with Laplace transform:

$$
U(s) := \mathcal{L}\{u(t)\} = \sum_{k=0}^{\infty} u(kT)\frac{1 - e^{-Ts}}{s}e^{-kTs}.
$$

We can therefore derive the discrete-time transfer function $G(z)$ (equivalent at the sample times to the continuous function $G(s)$ driven by a ZOH) by working out the impulse response of $(1 - e^{Ts})G(s)/s$ in continuous-time, then sampling it, and then finding its z-transform, i.e.:

$$
G(z) = \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{1 - e^{-Ts}}{s} G(s) \right\} \Big|_{t=kT} \right\}.
$$

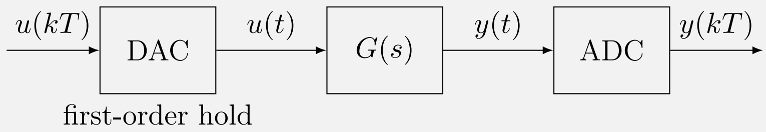
However e^{-Ts} represents a delay of T, which is equivalent to z^{-1} in the z-transform domain. Therefore we can simplify this to:

$$
G(z) = (1 - z^{-1})\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{G(s)}{s} \right\} \Big|_{t=kT} \right\}.
$$

♢

Question 10. Consider a continuous time system with transfer function $G(s)$ connected as shown in the figure below. The output of the first-order hold DAC is the linear extrapolation through the last two discrete inputs. Explain why the discrete system taking $\{u(kT)\}$ to $\{y(kT)\}$ has a z-transfer function. Show that the transfer function is given by the expression:

$$H(z) = \frac{(z-1)^2}{Tz^2} \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ G(s)\frac{Ts+1}{s^2} \right\}\Big|_{t=kT} \right\}$$



It is easy to check that the system satisfies linearity (indeed a linear scaling of an input produces the same effect on the output and adding two inputs adds the outputs) and time-invariance (shifiting the input by one time period T does the same to the ouptut). Hence, the system admits a z-transfer function. To compute the latter we can use an input $u(t)$ composed by the sum of an step function of amplitude T and a ramp:

☞ The choice of the input can be inferred from the factor $(z-1)^2/Tz^2$ in the expression of $H(z)$!

$$u(t) = T + t.$$

The Laplace transform of $u(t)$ is given by

$$U(s) := \mathcal{L}\{u(t)\} = \frac{T}{s} + \frac{1}{s^2} = \frac{Ts+1}{s^2}.$$

Hence the discrete-time output has the form

$$y(kT) = \mathcal{L}^{-1}\left\{ G(s)\frac{Ts+1}{s^2} \right\}\Big|_{t=kT},$$

and its z-transform is

$$Y(z) := \mathcal{Z}\{y(kT)\} = \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ G(s)\frac{Ts+1}{s^2} \right\}\Big|_{t=kT} \right\}.$$

Since $u(kT) = T + kT$, we have

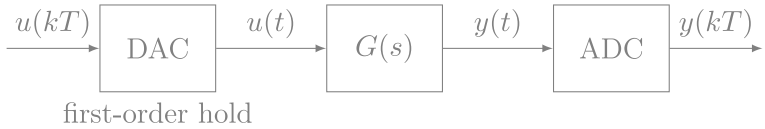
$$U(z) := \mathcal{Z}\{u(kT)\} = \frac{T}{1-z^{-1}} + \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{Tz^2}{(z-1)^2}.$$

Finally the transfer function of the overall system is given by

$$H(z) := \frac{Y(z)}{U(z)} = \frac{(z-1)^2}{Tz^2} \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ G(s)\frac{Ts+1}{s^2} \right\}\Big|_{t=kT} \right\}.$$

A more rigorous explanation of the fact that $H(z)$ is the transfer function of the discretised system is given below.

Consider the following block diagram



First consider the first-order hold (for short FOH). In the time domain, the output of the FOH can be written as

$$
\begin{aligned}
u(t) = \sum_{k=0}^{\infty} & u(kT)\delta_{-1}(t-kT) - 2u(kT)\delta_{-1}(t-(k+1)T) \\
& + u((k-1)T)\delta_{-1}(t-(k+1)T) + \frac{1}{T}\delta_1(t-kT)(u(kT)-u((k-1)T)) \\
& - \frac{1}{T}\delta_1(t-(k+1)T)(u(kT)-u((k-1)T)),
\end{aligned}
$$

where $\delta_1(t)$ denotes the ramp function (i.e. given $t \in \mathbb{R}$, $\delta_1(t) = t$ if $t \geq 0$ and $\delta_1(t) = 0$ otherwise). By taking the Laplace transform of the previous expression $U(s) := \mathcal{L}\{u(t)\}$, after some rearrangements, we obtain

$$
\begin{aligned}
U(s) &= \sum_{k=0}^{\infty} u(kT)\frac{1-2e^{-Ts}+e^{-2Ts}}{s}e^{-kTs} + u(kT)\frac{1-2e^{-Ts}+e^{-2Ts}}{Ts^2}e^{-kTs} \\
&= \sum_{k=0}^{\infty} u(kT)\frac{(Ts+1)}{s^2}\frac{(1-e^{-Ts})^2}{T}e^{-kTs} =: \frac{(Ts+1)}{s^2}\frac{(1-e^{-Ts})^2}{T}U^*(e^{sT}),
\end{aligned}
$$

where

$$
U^*(e^{sT}) := \sum_{k=0}^{\infty} u(kT)e^{-kTs},
$$

is the discrete transform of the original input $\{u(kT)\}$ (to recover the z-transform just put $z = e^{sT}$). We can now derive the discrete-time transfer function $H(z)$ as follows

$$
H(z) = \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{(Ts+1)}{s^2}\frac{(1-e^{-Ts})^2}{T} \right\}\Bigg|_{t=kT} \right\}.
$$

Eventually, since e^{-Ts} represents a delay of T, which is equivalent to z^{-1} in the z-transform domain, we get

$$
H(z) = \frac{(z^2-1)^2}{Tz^2}\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ G(s)\frac{Ts+1}{s^2} \right\}\Bigg|_{t=kT} \right\}.
$$

$\diamondsuit$

Question 11. An economic indicator is measured once every quarter and it is desired to estimate the underlying trend in the face of seasonal fluctuations. Assume that the indicator at the k^{th} quarter, v_k, is related to the underlying trend, u_k, by

$$
v_k = u_k + w_k, \quad k \geq 0,
$$

where $u_k = a + bk$, and the seasonal variation, w_k, is a periodic function with period 4 and zero mean value, (i.e. $w_{4k+i} = w_i$ and $w_0 + w_1 + w_2 + w_3 = 0$).

(a) Show that

$$W(z) = \mathcal{Z}\{w_k\} = \frac{w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}}{1 - z^4} \quad (3)$$

$$= \frac{w_0 + (w_0 + w_1) z^{-1} - w_3 z^{-2}}{(1 + z^{-1})(1 + z^{-2})} \quad (4)$$

(b) The underlying trend is estimated by passing $\{v_k\}$ through a FIR filter giving

$$y_k = \frac{1}{8}(v_k + 2v_{k-1} + 2v_{k-2} + 2v_{k-3} + v_{k-4})$$

with error given by

$$e_k = y_k - u_{k-2}.$$

(i.e. y_k is supposed to be an estimate of u_{k-2}).

(i) Find the transfer function, $G(z)$, of this filter, and its zeros, and show that the poles of $W(z)$ are cancelled by zeros of $G(z)$.

(ii) Show that $(G(z) - z^{-2}) = \frac{1}{8}(1 - z^{-1})^2(1 + 4z^{-1} + z^{-2})$, and hence that its zeros cancel the poles of $U(z)$.

(iii) Hence show that the error, e_k, will be zero for $k \geq 4$, and hence this filter accurately extracts the trend under these assumptions.

(iv) Show that

$$G(e^{j\theta}) = \cos\theta \cos^2(\theta/2) e^{-j2\theta}$$

and sketch the Bode diagram. Comment on the frequencies s.t. $G(e^{j\theta}) = 0$.

(a) The signal w_k is periodic of period 4, hence

$$\begin{aligned}
W(z) &= (w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&\quad + z^{-4}(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&\quad \vdots \\
&\quad + z^{-4k}(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&\quad \vdots \\
&= (1 + z^{-4} + z^{-8} + \cdots)(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&= \frac{1}{1 - z^{-4}}(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&= \frac{w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}}{(1 - z^{-1})(1 + z^{-1})(1 + z^{-2})} \\
&= \frac{w_0 + (w_0 + w_1) z^{-1} - w_3 z^{-2}}{(1 + z^{-1})(1 + z^{-2})},
\end{aligned}$$

where we used the fact that, since $w_2 = -w_0 - w_1 - w_3$,

$$w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3} = (1 - z^{-1})(w_0 + (w_0 + w_1) z^{-1} - w_3 z^{-2}).$$

(b) (i) The transfer function of the filter is

$$
\begin{aligned}
G(z) &= \frac{1}{8}(1 + 2z^{-1} + 2z^{-2} + 2z^{-3} + z^{-4}) \\
&= \frac{1}{8}(1 + z^{-1})(1 + z^{-1} + z^{-2} + z^{-3}) \\
&= \frac{1}{8}(1 + z^{-1})^2(1 + z^{-2}).
\end{aligned}
$$

We notice that $G(z)$ cancels the poles of $W(z)$.

(ii) The z-transform of u_k is given by

$$
U(z) := \mathcal{Z}\{u_k\} = \frac{a}{1 - z^{-1}} + \frac{bz^{-1}}{(1 - z^{-1})^2} = \frac{a(1 + z^{-1}) + bz^{-1}}{(1 - z^{-1})^2}.
$$

Now, we have

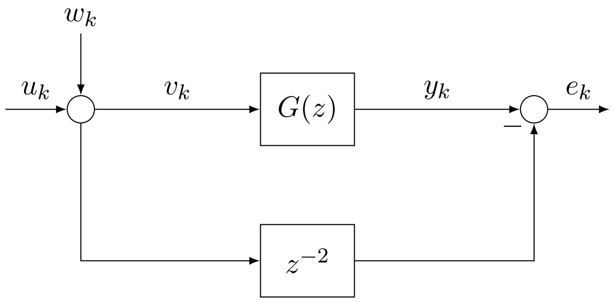
$$
\begin{aligned}
G(z) - z^{-2} &= \frac{1}{8}(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}) \\
&= \frac{1}{8}(1 - z^{-1})(1 + 3z^{-1} - 3z^{-2} + z^{-3}) \\
&= \frac{1}{8}(1 - z^{-1})^2(1 + 4z^{-1} + z^{-2}).
\end{aligned}
$$

Therefore the poles of $U(z)$ are cancelled by zeros of $(G(z) - z^{-2})$.

(iii) The z-transform of the error e_k is given by

$$
\begin{aligned}
E(z) := \mathcal{Z}\{e_k\} &= -z^2U(z) + G(z)(W(z) + U(z)) \\
&= (G(z) - z^{-2})U(z) + G(z)W(z) \\
&= \frac{1}{8}(1 + 4z^{-1} + z^{-2})(a(1 + z^{-1}) + bz^{-1}) \\
&\quad + \frac{1}{8}(1 + z^{-1})\left(w_0 + (w_0 + w_1)z^{-1} - w_3z^{-2}\right).
\end{aligned}
$$

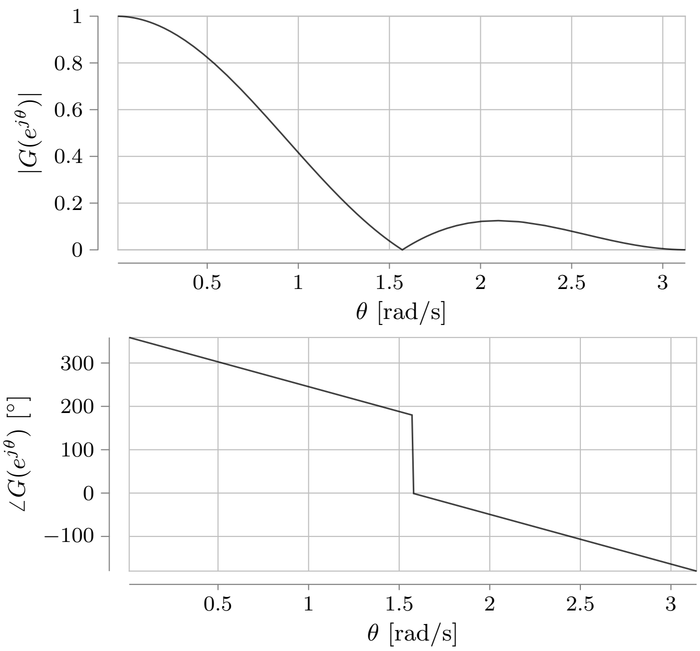
$E(z)$ is a polynomial in z^{-1} of degree 3, hence $e_4 = 0$ for $k \geq 4$. This in turn implies that $y_k = e_k + u_{k-2} = u_{k-2}$ for $k \geq 4$ and hence y_k gives the trend two periods ago.



(iv)

$$
\begin{aligned}
G(e^{j\theta}) &= \frac{1}{8}(1 + e^{-j\theta})^2(1 + e^{-2j\theta}) \\
&= \frac{1}{8}e^{-2j\theta}(e^{j\theta/2} + e^{-j\theta/2})^2(e^{j\theta} + e^{-j\theta}) \\
&= \cos\theta\cos^2(\theta/2)e^{-j2\theta}.
\end{aligned}
$$

$G(e^{j\theta}) = 0$ for $\theta = \pi/2, \pi$. The magnitude and phase Bode plots are showed below. In view of the equivalence $1/T = \theta/2\pi$, with T being the period, we infer that the zeros reject the period-4 periodic signals (such as w_k) and the period-2 periodic signals.



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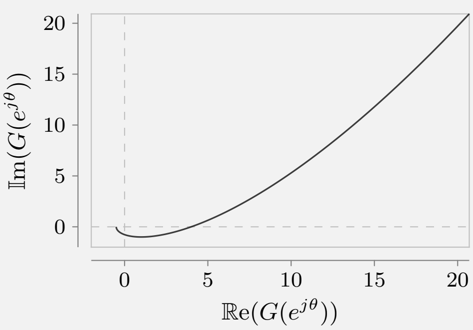
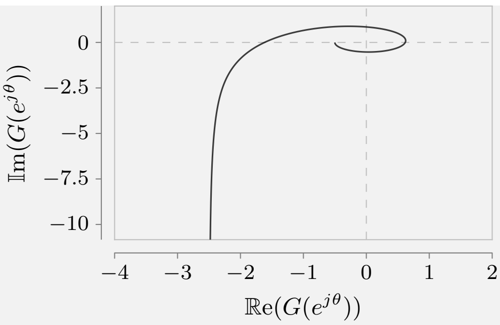
Question 12. The plots of $G(e^{j\theta})$ for θ ranging from 0^+ to $+\infty$ are shown below (in random order) for the following transfer functions:

$$\text{(i)} \quad \frac{1}{z^2(z-1)}, \qquad \text{(ii)} \quad \frac{4z+2}{3(z-1)^2}, \qquad \text{(iii)} \quad \frac{4}{(z-1)^3}.$$

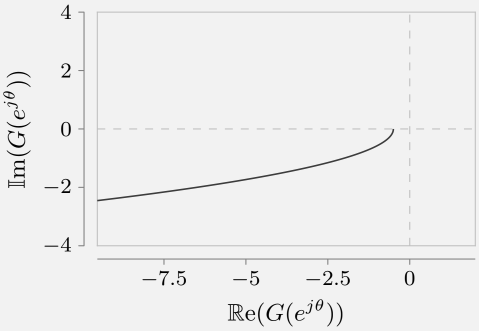
Sketch the complete Nyquist diagrams for each transfer function and use the Nyquist criterion to determine for what values of gain (if any) closed loop stability will be achieved when constant gain negative feedback is connected around them.



(a)



(b)

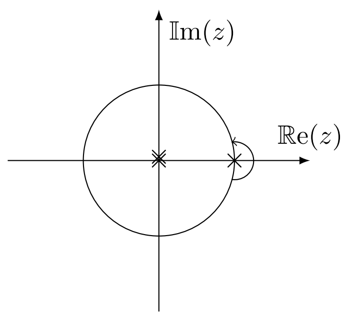


(c)

Consider the transfer function (i), namely

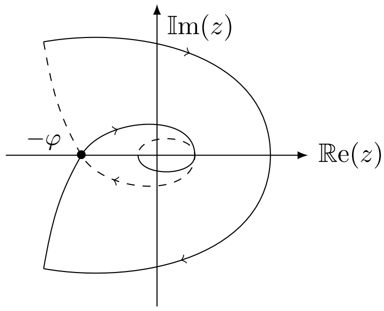
$$G_1(z) := \frac{1}{z^2(z-1)}.$$

We see that the phase plot ranging from $\theta = 0^+$ to $\theta = \pi$ goes from $-90°$, due to the pole at $z = 1$, to $-540°$, due to the contribution of the pole at $z = 1$ $(-180°)$ and of the two poles at $z = 0$ $(-360°)$. This corresponds to diagram (b). Since there is a pole at $z = 1$, we have to indent the path of z around $z = 1$ with a small semi-circular excursion outside the unit circle, as showed in the plot below.



The complete Nyquist plot is reported below. Since the system has no open-loop unstable poles, then, by virtue of the Nyquist stability criterion, the closed-loop system is stable if (and only if) the point $-1/K$ has no (counterclockwise) encirclements.

⚠ Recall that the poles on the unit circle are considered as stable poles in the Nyquist stability criterion!



To find the value of K that guarantee closed-loop stability we have to compute the first cross of the real axis of the Nyquist plot. To this aim, we first compute the frequency $\theta^* \neq \pi$ which satisfies the phase condition

An alternative way to compute θ^* is to use the condition $\mathbb{I}\mathrm{m}G(e^{j\theta^*}) = 0$.

$$\angle G_1(e^{j\theta^*}) = -\pi.$$

Since $\tan\left(\frac{\theta}{2} + \frac{\pi}{2}\right) = 1/\tan\left(-\frac{\theta}{2}\right) = \frac{\sin\theta}{\cos\theta - 1}$, we have

Define $t := \tan\frac{\theta}{2}$ and make the substitutions $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$.

$$\begin{aligned}
\angle G_1(e^{j\theta}) &= -\angle e^{j2\theta} - \angle(e^{j\theta} - 1) \\
&= -2\theta - \arctan\left(\frac{\sin\theta}{\cos\theta - 1}\right) \\
&= -2\theta - \frac{\theta}{2} - \frac{\pi}{2} \overset{!}{=} -\pi,
\end{aligned}$$

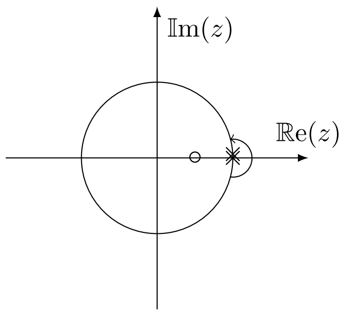
from which follows that $\theta^* = \pi/5$. By computing $G(e^{j\theta^*}) = -\frac{1+\sqrt{5}}{2} = -\varphi$ we conclude that the first cross of the real axis is at $z = -\varphi$. Eventually the closed system is stable if (and only if)

$$-\frac{1}{K} < -\varphi \quad \Rightarrow \quad 0 < K < \frac{1}{\varphi}.$$

Consider now the transfer function (ii), namely

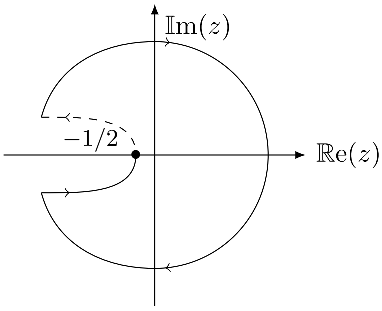
$$G_2(z) := \frac{4z + 2}{3(z-1)^2}.$$

The phase plot is close to $-180°$ for $\theta = 0^+$ (due to the double pole at $z = 1$) and goes back to $180°$ for $\theta = \pi$ (due to the poles and the zero at $z = 1/2$). This corresponds to diagram (c).



The below figure shows the complete Nyquist diagram. Since the system has no open-loop unstable poles, then no (counterclockwise) encirclements of $-1/K$ are required for closed-loop stability. Hence we set

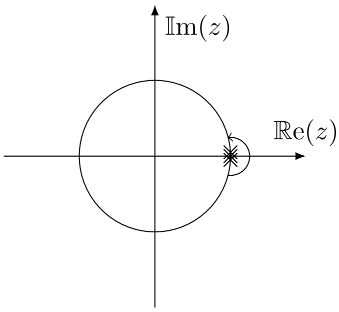
$$-\frac{1}{K} < -\frac{1}{2} \quad \Rightarrow \quad 0 < K < 2.$$



Finally, consider the transfer function (iii), namely

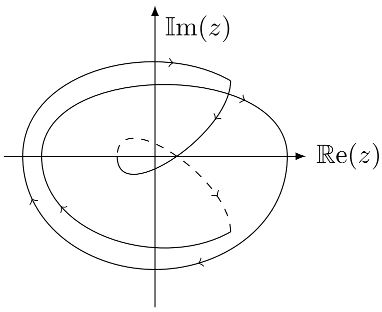
$$G_3(z) := \frac{4}{(z-1)^3}.$$

The phase of the transfer function is initially close to $-270°$ for $\theta = 0^+$ and decreases to $-540°$ for $\theta = \pi$ (due to the triple pole at $z = 1$) . This is diagram (a).



The complete Nyquist diagram is reported below. Since the system has no open-loop unstable poles, then no (counterclockwise) encirclements are required for closed-loop stability. But since all points of the real axis are encircled the closed-loop system is unstable for all $K > 0$.

☞ Notice that the large 540° circular arc accounts for the fact that there are 3 poles on the unit circle.



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