Stochastic processes: a behavioral perspective

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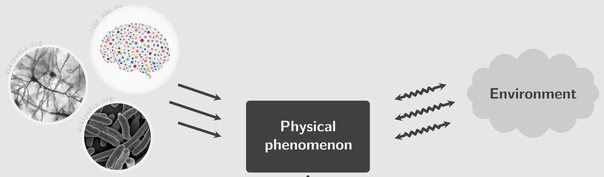
DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

Motivations



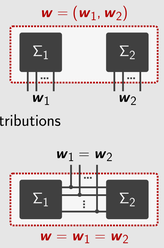
Dynamical
- Continuous time
- Discrete time
- Hybrid

Stochastic
- Gaussian
- Poisson
- ...

Open
Interacting with environment ⇕ Interconnectable

Motivating questions:

1. How to define ***interconnection*** for stochastic systems?
   - Interconnection as **coupling** ≈ Modelling of probability distributions
   - Interconnection as **variable sharing** ≈ Modelling of event spaces



2. What is the ***rôle of noise*** in biology?
   - Noise as an unavoidable **nuisance**
   - Noise as a **benefit** → Synchronization, Robustness, Connecting different scales

Behavioral systems theory: a framework for modelling open systems

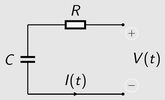
dynamical system

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

time axis | signal space | behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$

set of admissible trajectories

electrical example:



$\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^2$
$w = (V, I)$
$\mathfrak{B} = \{(V, I) : R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}\}$

Interconnection

Interconnection = variable sharing

$\Sigma_1 \wedge \Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_1 \cap \mathfrak{B}_2)$

LTI systems

Σ is **Linear and Time-Invariant** if:

1. $\alpha w_1 + \beta w_2 \in \mathfrak{B}$, $\forall\, w_1, w_2 \in \mathfrak{B}$, $\forall\, \alpha, \beta$ scalars of the vector space $\mathbb{W}^{\mathbb{T}}$
2. $\sigma w, \sigma^{-1} w \in \mathfrak{B}$ $\forall\, w \in \mathfrak{B}$, σ, σ^{-1} = forward, backward time-shift operator

Σ **complete** if uniquely determined by its finite dimensional restrictions:

$$w \in \mathfrak{B} \iff w|_{[t_1,t_2]} \in \mathfrak{B}|_{[t_1,t_2]}, \quad \forall\, t_1, t_2 \in \mathbb{T}$$

Stochastic processes from a "behavioral" viewpoint

stochastic dynamical system

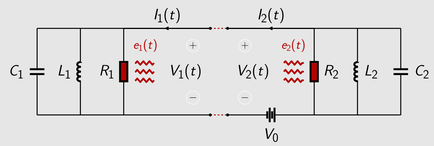
noise acting on the trajectories of Σ

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{E}, P)$$

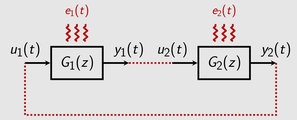
time axis | signal space | event σ-algebra on $\mathbb{W}^{\mathbb{T}}$ ≈ | probability measure

collection of admissible behaviors "weighted" by P

Interconnection of stochastic processes

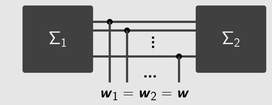


$V_1(t) = V_2(t)$, $I_1(t) = I_2(t)$



$y_1(t) = u_2(t)$, $y_2(t) = u_1(t)$

$\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathcal{E}_1, P_1)$



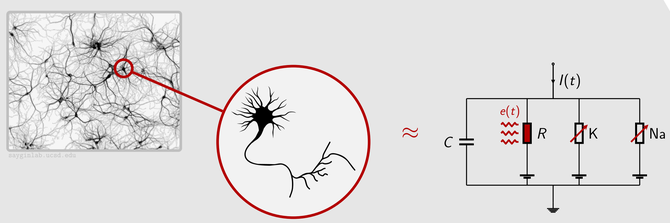
$\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathcal{E}_2, P_2)$

$\Sigma_1 \wedge \Sigma_2 = (\mathbb{W}, \mathbb{T}, \mathcal{E}, P)$

- $\mathcal{E}$ = σ-algebra generated by the sets $E_1 \cap E_2$, $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$
- P defined by $P(E_1 \cap E_2) = P_1(E_1)P_2(E_2)$ and extended to all of $\mathcal{E}$

Interconnection = variable sharing + compatibility of probability laws

Potential applications to neuroscience



Hodgkin-Huxley model + noise

interconnection of neurons ≈ interconnection of stochastic processes

Summary and future directions

- Behavioral theory [1] offers a mathematical framework for modelling ***open*** systems. In this framework interconnection is seen as ***variable sharing*** between two systems.
- Following [2], we analyze an extension of behavioral theory to stochastic dynamical systems and its implications in terms of ***interconnection*** and ***invariance properties***.

Future research directions include the analysis of continuous-time stochastic processes and potential applications to interconnected biological systems affected by noise.

Invariance properties in the space of LTI stochastic processes

Static case

$$Rw = e$$

e = n-dim. Gaussian vector $\mathcal{N}(\mu, \Gamma)$

Equivalence class:

$R \longrightarrow UR$
$\mu \longrightarrow U\mu$
$\Gamma \longrightarrow U\Gamma U^\top$

$U \in \mathbb{GL}(n)$

$\{U \in \mathbb{R}^{n\times n} : \det(U) \neq 0\}$

Affine invariance of covariance matrices

Dynamic case

$$R(\sigma, \sigma^{-1})w(t) = e(t)$$

$e(t)$ = n-dim. Gaussian process $\mathcal{GP}(\mu(t), \Gamma(t,s))$

Equivalence class:

$R \longrightarrow UR$
$\mu(t) \longrightarrow U\mu(t)$
$\Gamma(t,s) \longrightarrow U\Gamma(t,s)U^\top$

$U \in \mathbb{U}[\sigma, \sigma^{-1}]^{n\times n}$

$\{U \in \mathbb{R}[\sigma, \sigma^{-1}]^{n\times n} : \det(U) = \text{cost}\}$

Unimodular invariance of covariance functions

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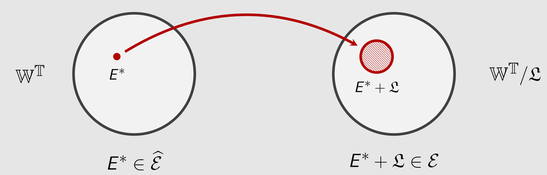
References:

[1] J.C. Willems, J.W. Polderman. *Introduction to mathematical systems theory: a behavioral approach.* Springer, 2013.
[2] J.C. Willems "Open stochastic systems." IEEE Transactions on Automatic Control, 58 **2** (2013): 406-421.

LTI stochastic processes

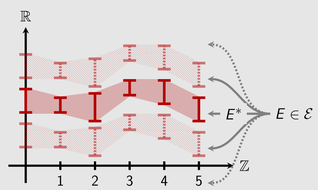
$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{E}, P)$ is **Linear and Time-Invariant** if the event σ-algebra $\mathcal{E}$ is built on the space of trajectories $\mathbb{W}^{\mathbb{T}}$ modulo a LTI behavior $\mathfrak{L}$



Discrete-time LTI processes

- $\mathbb{T} = \mathbb{Z}$
- $\mathbb{W} = \mathbb{R}^n$
- $\hat{\mathcal{E}} = \mathcal{B}(\mathbb{R}^n)^{\mathbb{Z}}$ = Borel σ-algebra of $(\mathbb{R}^n)^{\mathbb{Z}}$ equipped with the product topology

1-D example:



$\mathfrak{L} = \{w \in \mathbb{R}^{\mathbb{Z}} : w(t+1) = w(t)\}$

Canonical AR representation:

$\mathfrak{L}$ LTI and complete

⇕

$$\underset{1}{R(\sigma, \sigma^{-1})}\ \underset{2}{w(t)} = \underset{3}{e(t)}$$

1. $m \times n$ Laurent polynomial matrix = linear operator in the shift σ
2. n-dim. time sequence = stochastic law of the process
3. m-dim. process $\mathcal{E}_e = \mathcal{B}(\mathbb{R}^m)^{\mathbb{Z}}$ = exogenous source of noise