The Controllability Gramian of Line Networks: Closed-Form Expressions and Asymptotic Transitions

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***Abstract*— In this technical note, we establish closed-form expressions of the entries of the (output) controllability Gramian of a class of bidirectional line networks. Also, we characterize the asymptotic behavior of these entries in two important cases.**

I. Problem formulation

We consider networks governed by linear time-invariant continuous-time dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ denote the vectors of nodes' states, inputs, and outputs at time t, respectively. The matrix $A \in \mathbb{R}^{n\times n}$ denotes the (weighted and directed) adjacency matrix of the network, and $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times m}$ are the input and output matrices, respectively. These matrices are chosen so as to single out prescribed sets of input and output nodes of the network, that is,

$$B = \begin{bmatrix} \mathrm{e}_{k_1} & \cdots & \mathrm{e}_{k_m} \end{bmatrix}, \quad C = \begin{bmatrix} \mathrm{e}_{t_1} & \cdots & \mathrm{e}_{t_p} \end{bmatrix}^\top, \tag{2}$$

where $\mathcal{K} = \{k_1, k_2, \dots, k_m\}$ and $\mathcal{T} = \{t_1, t_2, \dots, t_p\}$ are the sets of input and output nodes, respectively, and $\{\mathrm{e}_i\}_{i=1}^n$ denote the vectors of the canonical basis of $\mathbb{R}^n$.

If A is Hurwitz stable, the infinite-horizon output controllability Gramian of (1) is well-defined and given by

$$\mathcal{W} = \int_0^\infty C e^{At} B B^\top e^{A^\top t} C^\top \mathrm{d}t. \tag{3}$$

The (output) controllability Gramian is linked to the controllability properties of the network, in that its eigenvalues describe how much control energy is needed to reach different output directions using a minimum-norm control input [].

In this note, we analyze the output controllability Gramian of a simple yet insightful class of networks. Namely, we consider bidirectional line networks which are described by the following Toeplitz adjacency matrix

$$A = \begin{bmatrix} \gamma & \beta/\alpha & 0 & \cdots & 0 \\ \beta\alpha & \gamma & \beta/\alpha & & \vdots \\ 0 & \beta\alpha & \gamma & \ddots & 0 \\ \vdots & & \ddots & \ddots & \beta/\alpha \\ 0 & \cdots & 0 & \beta\alpha & \gamma \end{bmatrix}, \tag{4}$$

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where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are positive parameters and $\gamma \in \mathbb{R}$ is chosen such that $\gamma < -2\beta$ so as to enforce stability. Notice that the parameter α quantifies, in a sense, the "degree" of directionality of the network. Indeed, the larger α the stronger is the connection from node i to node $i+1$ and the weaker is the connection in the opposite direction. Thus, the network in (4) represents a simple, prototypical architecture in which the effects of directionality (or, in algebraic terms, non-normality) and stability are completely decoupled and can be freely tuned. More precisely, the directionality is regulated by parameter α, whereas the eigenvalues are determined by parameters β and γ. Finally, for later use, we observe that A can be rewritten as

$$A = DSD^{-1}, \tag{5}$$

where

$$S = \begin{bmatrix} \gamma & \beta & 0 & \cdots & 0 \\ \beta & \gamma & \beta & & \vdots \\ 0 & \beta & \gamma & \ddots & 0 \\ \vdots & & \ddots & \ddots & \beta \\ 0 & \cdots & 0 & \beta & \gamma \end{bmatrix}, \tag{6}$$

is a symmetric matrix featuring the same spectrum of A, and $D = \mathrm{diag}[1 \ \alpha \ \alpha^2 \ \cdots \ \alpha^{n-1}]$ a diagonal matrix whose diagonal encodes the degree of directionality of the network.

II. Finite-size analysis of $\mathcal{W}$

In this section, we establish a closed-form expression of the controllability Gramian (3).

Theorem 1: ***(Closed-form expression of*** $\mathcal{W}$***)*** Consider the output controllability Gramian (3) where A is as in (4), and B and C are as in (2). For all $i, j \in \{1, \dots, p\}$, it holds

$$[\mathcal{W}]_{ij} = -\frac{2}{N^2} \sum_{k\in\mathcal{K}} \sum_{\ell=1}^{N-1} \sum_{h=1}^{N-1} \frac{\alpha^{t_i+t_j-2k}}{\gamma + \beta(\cos(x_\ell) + \cos(x_h))} \cdot \sin(t_i x_\ell) \sin(k x_\ell) \sin(t_j x_h) \sin(k x_h), \tag{7}$$

where $x_i := i\pi/N$, $i = 1, \dots, N-1$, and $N := n+1$.

Before presenting the proof of Theorem (1), we state an instrumental lemma, whose proof can be found in, e.g., [1, Ex. 7.2.5].

Lemma 2: ***(Eigenvalues and eigenvectors of*** S***)*** The matrix S as defined in (6) admits the spectral decomposition

$$S = V^\top \Lambda V, \tag{8}$$

where $\Lambda = \mathrm{diag}[\lambda_1 \ \cdots \ \lambda_n]$ is a diagonal matrix containing the eigenvalues of S

$$\lambda_k = \gamma + 2\beta \cos\left(\frac{k\pi}{n+1}\right), \quad k \in \{1, \dots, n\}, \qquad (9)$$

and the columns of $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ the corresponding (normalized) eigenvectors

$$v_k = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin\left(\frac{k\pi}{n+1}\right) \\ \sin\left(\frac{2k\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nk\pi}{n+1}\right) \end{bmatrix}, \quad k \in \{1, \dots, n\}. \qquad (10)$$

Proof of Theorem 1: In view of the definition of B in (2), it follows that $BB^\top = \sum_{k\in\mathcal{K}} \mathrm{e}_k \mathrm{e}_k^\top$. Thus, we can rewrite $\mathcal{W}$ as

$$\begin{aligned} \mathcal{W} &= \int_0^\infty C e^{At} BB^\top e^{A^\top t} C^\top \mathrm{d}t \\ &= \int_0^\infty C e^{At} \left(\sum_{k\in\mathcal{K}} \mathrm{e}_k \mathrm{e}_k^\top\right) e^{A^\top t} C^\top \mathrm{d}t \\ &= \sum_{k\in\mathcal{K}} \int_0^\infty C e^{At} \mathrm{e}_k \mathrm{e}_k^\top e^{A^\top t} C^\top \mathrm{d}t. \end{aligned} \qquad (11)$$

From the definition of C in (2), the (i,j)-th entry of $\mathcal{W}$ reads

$$[\mathcal{W}]_{ij} = \sum_{k\in\mathcal{K}} \mathrm{e}_{t_i}^\top \left(\int_0^\infty e^{At} \mathrm{e}_k \mathrm{e}_k^\top e^{A^\top t} \mathrm{d}t\right) \mathrm{e}_{t_j}. \qquad (12)$$

Next, by using the decomposition of A in (5), we have

$$\begin{aligned} [\mathcal{W}]_{ij} &= \sum_{k\in\mathcal{K}} \mathrm{e}_{t_i}^\top \left(\int_0^\infty D e^{St} D^{-1} \mathrm{e}_k \mathrm{e}_k^\top D^{-1} e^{St} D \mathrm{d}t\right) \mathrm{e}_{t_j} \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\alpha^{2(k-1)}} \mathrm{e}_{t_i}^\top D \left(\int_0^\infty e^{St} \mathrm{e}_k \mathrm{e}_k^\top e^{St} \mathrm{d}t\right) D \mathrm{e}_{t_j} \\ &= \sum_{k\in\mathcal{K}} \frac{\alpha^{t_i+t_j-2}}{\alpha^{2(k-1)}} \left(\int_0^\infty \mathrm{e}_{t_i}^\top e^{St} \mathrm{e}_k \mathrm{e}_k^\top e^{St} \mathrm{e}_{t_j} \mathrm{d}t\right). \end{aligned} \qquad (13)$$

Now, we focus on the integral terms in (13), that is,

$$I_{ijk} = \int_0^\infty \mathrm{e}_{t_i}^\top e^{St} \mathrm{e}_k \mathrm{e}_k^\top e^{St} \mathrm{e}_{t_j} \mathrm{d}t. \qquad (14)$$

By Lemma 2, it holds

$$\begin{aligned} I_{ijk} &= \int_0^\infty \mathrm{e}_{t_i}^\top V^\top e^{\Lambda t} V \mathrm{e}_k \mathrm{e}_k^\top V^\top e^{\Lambda t} V \mathrm{e}_{t_j} \mathrm{d}t \\ &= \int_0^\infty v_{t_i}^\top e^{\Lambda t} v_k v_k^\top e^{\Lambda t} v_{t_j} \mathrm{d}t. \end{aligned} \qquad (15)$$

Note that, by direct computation,

$$\begin{aligned} v_h^\top e^{\Lambda t} v_k &= \frac{2}{n+1} \cdot \\ &\cdot \begin{bmatrix} \sin\left(\frac{h\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nh\pi}{n+1}\right) \end{bmatrix}^\top \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_1 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \sin\left(\frac{k\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nk\pi}{n+1}\right) \end{bmatrix} \\ &= \frac{2}{n+1} \sum_{\ell=1}^n e^{\lambda_\ell t} \sin\left(\frac{\ell h\pi}{n+1}\right) \sin\left(\frac{\ell k\pi}{n+1}\right), \end{aligned} \qquad (16)$$

which plugged into (15) yields

$$\begin{aligned} I_{ijk} &= \int_0^\infty v_{t_i}^\top e^{\Lambda t} v_k v_k^\top e^{\Lambda t} v_{t_j} \mathrm{d}t \\ &= \frac{4}{(n+1)^2} \sum_{\ell=1}^n \sum_{h=1}^n \sin\left(\frac{\ell t_i \pi}{n+1}\right) \sin\left(\frac{\ell k\pi}{n+1}\right) \cdot \\ &\quad \cdot \sin\left(\frac{h t_j \pi}{n+1}\right) \sin\left(\frac{hk\pi}{n+1}\right) \int_0^\infty e^{(\lambda_\ell+\lambda_h)t} \mathrm{d}t \\ &= -\frac{4}{(n+1)^2} \sum_{\ell=1}^n \sum_{h=1}^n \frac{1}{\lambda_h+\lambda_\ell} \sin\left(\frac{\ell t_i \pi}{n+1}\right) \cdot \\ &\quad \cdot \sin\left(\frac{\ell k\pi}{n+1}\right) \sin\left(\frac{h t_j \pi}{n+1}\right) \sin\left(\frac{hk\pi}{n+1}\right) \\ &= -\frac{2}{(n+1)^2} \sum_{\ell=1}^n \sum_{h=1}^n \frac{1}{\gamma+\beta\left(\cos\left(\frac{\ell\pi}{n+1}\right)+\cos\left(\frac{h\pi}{n+1}\right)\right)} \cdot \\ &\cdot \sin\left(\frac{\ell t_i \pi}{n+1}\right) \sin\left(\frac{\ell k\pi}{n+1}\right) \sin\left(\frac{h t_j \pi}{n+1}\right) \sin\left(\frac{hk\pi}{n+1}\right), \end{aligned} \qquad (17)$$

where in the second step we used the fact that $\int_0^\infty e^{(\lambda_\ell+\lambda_h)t} = \frac{1}{\lambda_\ell+\lambda_h}$, and in the last step the analytic expression of λ_k, $k \in \{1, \dots, n\}$, in Lemma 2. Finally, equation (7) follows by substituting (17) into (13). ■

An interesting scenario is when the input signal enters the network from the first node of the network ($\mathcal{K} = \{1\}$). In this case, two extreme input/output configurations are when the input and output nodes coincide ($\mathcal{K} = \{1\}$ and $\mathcal{T} = \{1\}$), and when they are placed as far away as possible ($\mathcal{K} = \{1\}$ and $\mathcal{T} = \{n\}$). In these two extreme cases, it is possible to establish simplified versions of the expressions in Theorem 1.

Corollary 3: ***(Closed-form expression of*** $\mathcal{W}$ ***for*** $\mathcal{K} = \{1\}$ ***and*** $\mathcal{T} = \{1\}$***)*** Consider the output controllability Gramian (3) where A is as in (4), and B and C are as in (2). Further, let $x_i := i\pi/N$, $i = 1, \dots, N-1$, and $N := n+1$. If $\mathcal{K} = \{1\}$ and $\mathcal{T} = \{1\}$, then it holds

$$\mathcal{W} = -\frac{2}{N^2} \sum_{\ell=1}^{N-1} \sum_{h=1}^{N-1} \frac{\sin^2(x_\ell)\sin^2(x_h)}{\gamma+\beta\left(\cos(x_\ell)+\cos(x_h)\right)}. \qquad (18)$$

Proof: Equation (18) directly follows by substituting $k = t_i = t_j = 1$ in (7). ■

Corollary 4: ***(Closed-form expression of*** $\mathcal{W}$ ***for*** $\mathcal{K} = \{1\}$ ***and*** $\mathcal{T} = \{n\}$***)*** Consider the output controllability Gramian (3) where A is as in (4), and B and C are as in (2). Further, let $x_i := i\pi/N$, $i = 1, \dots, N-1$, and $N := n+1$. If $\mathcal{K} = \{1\}$ and $\mathcal{T} = \{n\}$, then it holds

$$\mathcal{W} = -\frac{2\alpha^{2(N-2)}}{N^2} \sum_{\ell=1}^{N-1} \sum_{h=1}^{N-1} \frac{(-1)^{\ell+h} \sin^2(x_\ell) \sin^2(x_h)}{\gamma + \beta(\cos(x_\ell) + \cos(x_h))}. \tag{19}$$

Proof: By letting $k = 1$ and $t_i = t_j = n$, equation (7) takes the form

$$\mathcal{W} = -\frac{2}{N^2} \sum_{\ell=1}^{N-1} \sum_{h=1}^{N-1} \frac{\alpha^{2(n-1)}}{\gamma + \beta\left(\cos\left(\frac{\ell\pi}{N}\right) + \cos\left(\frac{h\pi}{N}\right)\right)} \cdot \sin\left(\frac{\ell n\pi}{N}\right) \sin\left(\frac{\ell\pi}{N}\right) \sin\left(\frac{hn\pi}{N}\right) \sin\left(\frac{h\pi}{N}\right), \tag{20}$$

and equation (19) follows from (20) by using the identity

$$\sin\left(\frac{qn\pi}{N}\right) = \sin\left(-\frac{q\pi}{n+1} + q\pi\right) = (-1)^{q+1} \sin\left(\frac{q\pi}{N}\right), \quad q \in \mathbb{Z}.$$

■

III. Asymptotic analysis of $\mathcal{W}$

In this section, we study the large n asymptotic behavior of the controllability Gramian (3) for the line network in (4) and the two extreme scenarios discussed in Corollaries 3 and 4, that is, when the input and output nodes coincide ($\mathcal{K} = \{1\}$ and $\mathcal{T} = \{1\}$), and when are placed as far away as possible ($\mathcal{K} = \{1\}$ and $\mathcal{T} = \{n\}$).

Theorem 5: ***(Asymptotic behavior for $\mathcal{K} = \mathcal{T} = \{1\}$)*** Consider the output controllability Gramian (3) where A is as in (4), and B and C are as in (2). If $\mathcal{K} = \{1\}$ and $\mathcal{T} = \{1\}$, then as $n \to \infty$, $\mathcal{W}$ converges to a positive constant satisfying

$$\frac{\pi^2}{-2\gamma + 4\beta} \le \mathcal{W} \le \frac{\pi^2}{-2\gamma - 4\beta}. \tag{21}$$

Proof: Note that (18) can be equivalently written as

$$\mathcal{W} = -\frac{2}{N^2} \sum_{\ell=0}^{N-1} \sum_{h=0}^{N-1} \frac{\sin^2(x_\ell) \sin^2(x_h)}{\gamma + \beta(\cos(x_\ell) + \cos(x_h))}, \tag{22}$$

where we used the fact that the terms in the summation corresponding to the indices $\ell = 0$ and $h = 0$ vanish. In the limit $n \to \infty$, equation (22) converges to the integral

$$\mathcal{W} = -2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\sin^2(x) \sin^2(y)}{\gamma + \beta(\cos(x) + \cos(y))} \, \mathrm{d}x \, \mathrm{d}y. \tag{23}$$

Since $-\gamma - 2\beta \le \gamma + \beta(\cos(x) + \cos(y)) \le -\gamma + 2\beta$, we can bound the integral (23) as

$$\frac{2I}{-\gamma + 2\beta} \le \mathcal{W} \le \frac{2I}{-\gamma - 2\beta}, \tag{24}$$

with $I := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sin^2(x) \sin^2(y) \, \mathrm{d}x \, \mathrm{d}y = \pi^2/4$, from which (21) follows. ■

When $\mathcal{K} = \mathcal{T} = \{1\}$, Corollary 5 guarantees that the Gramian is always bounded and independent of n. Further, for very stable networks (large $|\gamma|$), the inequalities in (21) yields the estimate $\mathcal{W} \sim -\pi^2/(2\gamma)$.

Theorem 6: ***(Asymptotic behavior for $\mathcal{K} = \{1\}$ and $\mathcal{T} = \{n\}$)*** Consider the output controllability Gramian (3) where A is as in (4), and B and C are as in (2). If $\mathcal{K} = \{1\}$ and $\mathcal{T} = \{n\}$, then as $n \to \infty$ it holds

$$\mathcal{W} \sim \frac{\mu}{\sqrt{n}} \left(\alpha\left(\kappa - \sqrt{\kappa^2 - 1}\right)\right)^{2n}. \tag{25}$$

where $\kappa := -\gamma/(2\beta) > 1$ and $\mu > 0$ is a real constant independent of n and depending only on α, β and γ.

To prove Theorem 6, we will make use of the following lemma, that has been adapted from [2, Sec. 4(b)].

Lemma 7: Let $n > 0$ and $\kappa > 1$ be real numbers. Then, as $n \to \infty$,

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-\mathrm{i}n(x+y)}}{2\kappa - \cos(x) - \cos(y)} \, \mathrm{d}x \, \mathrm{d}y \sim \frac{\xi}{\sqrt{n}} \left(\kappa - \sqrt{\kappa^2 - 1}\right)^{2n}, \tag{26}$$

where $\xi := 1/(2\sqrt{\pi\kappa}(\kappa^2 - 1)^{1/4})$.

Proof of Theorem 6: Let $N := n+1$ and define

$$\Psi(x, y) := \frac{1}{\beta} \frac{\sin^2(2\pi x) \sin^2(2\pi y)}{2\kappa - \cos(2\pi x) - \cos(2\pi y)}. \tag{27}$$

In view of Corollary 4, we can write $\mathcal{W}$ as

$$\mathcal{W} = \frac{2\alpha^{2(N-2)}}{N^2} \sum_{\ell=1}^{N-1} \sum_{h=1}^{N-1} (-1)^{\ell+h} \Psi\left(\frac{\ell}{2N}, \frac{h}{2N}\right). \tag{28}$$

Notice that $\Psi(x, y) = \Psi(-x, y) = \Psi(x, -y) = \Psi(-x, -y)$ and $\Psi(0, y) = \Psi(1/2, y) = \Psi(x, 0) = \Psi(x, 1/2) = 0$. Therefore, we can rewrite (28) as

$$\mathcal{W} = \frac{\alpha^{2(N-2)}}{2N^2} \sum_{\ell=1}^{2N} \sum_{h=1}^{2N} (-1)^{\ell+h} \Psi\left(\frac{\ell}{2N}, \frac{h}{2N}\right). \tag{29}$$

The latter equation follows from the fact that each term in (28) appears four times in (29) and the additional terms corresponding to indices $\ell, h \in \{N, 2N\}$ vanish. Next, we can express it in terms of the 2D Fourier series

$$\Psi(x, y) := \sum_{r,s \in \mathbb{Z}} \psi_{r,s} e^{2\pi\mathrm{i}(rx+sy)}, \tag{30}$$

which converges absolutely since $\Psi(x, y)$ is smooth, and substitute the latter series in (29). By doing so, we obtain

$$\begin{aligned} \mathcal{W} &= \frac{\alpha^{2(N-2)}}{2N^2} \sum_{\ell=1}^{2N} \sum_{h=1}^{2N} (-1)^{\ell+h} \sum_{r,s \in \mathbb{Z}} \psi_{r,s} e^{2\pi\mathrm{i}\left(\frac{r\ell}{2N} + \frac{sh}{2N}\right)} \\ &= \frac{\alpha^{2(N-2)}}{2N^2} \sum_{r,s \in \mathbb{Z}} \psi_{r,s} \sum_{\ell=1}^{2N} (-1)^\ell e^{2\pi\mathrm{i}\frac{r\ell}{2N}} \sum_{h=1}^{2N} (-1)^h e^{2\pi\mathrm{i}\frac{sh}{2N}} \\ &= 2\alpha^{2(N-2)} \sum_{r,s \in \mathbb{Z}} \psi_{N(2r+1), N(2s+1)}, \end{aligned} \tag{31}$$

where in the last step we used the identity, $q \in \mathbb{Z}$,

$$\sum_{h=1}^{2N} (-1)^h e^{2\pi q\mathrm{i}\frac{h}{2N}} = \begin{cases} 2N, & \text{if } h = N \bmod 2N, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier coefficients in (30) read as

$$\psi_{N(2r+1),N(2s+1)} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \bar{\Psi}\left(x,y\right)e^{-\mathrm{i}N((2r+1)x+(2s+1)y)}\,\mathrm{d}x\,\mathrm{d}y, \quad (32)$$

where $\bar{\Psi}\left(x,y\right) := \Psi\left(x/2\pi, y/2\pi\right)$. Notice that the function $\bar{\Psi}(x,y)$ can be extended to a complex analytic function in the complex strip $\{x,y \in \mathbb{C} : |\mathrm{Im}(x)| \leq K,\ |\mathrm{Im}(y)| \leq K\}$, where $K := \cosh^{-1}(\kappa) = \ln\left(\kappa + \sqrt{\kappa^2-1}\right)$. Thus, as a consequence of the Paley–Wiener Theorem (e.g., see [3, §VI.7]), the Fourier coefficients in (32) decay exponentially with a rate that satisfies, for all $\varepsilon > 0$,

$$\begin{aligned}|\psi_{N(2r+1),N(2s+1)}| &\leq M(\varepsilon)e^{-2KN(r+s+1-\varepsilon)}\\ &\leq M(\varepsilon)\left(\kappa+\sqrt{\kappa^2-1}\right)^{-2N(r+s+1-\varepsilon)}\\ &\leq M(\varepsilon)\left(\kappa-\sqrt{\kappa^2-1}\right)^{2N(r+s+1-\varepsilon)}\end{aligned} \quad (33)$$

where $M(\varepsilon)$ is a positive real constant depending only on ε and in the last step we used the identity $\left(\kappa+\sqrt{\kappa^2-1}\right)^{-1} = \left(\kappa-\sqrt{\kappa^2-1}\right)$. We next show that the dominant (i.e., slowest decaying) coefficients are those corresponding to the "simplest" terms of the series (31), namely $\psi_{N,N}$, $\psi_{-N,N}$, $\psi_{N,-N}$, $\psi_{-N,-N}$. Since the Fourier coefficients satisfy $\psi_{r,s} = \psi_{-r,s} = \psi_{r,-s} = \psi_{-r,-s}$, the "simplest" four coefficients of the series (31) are all equal to $\psi_{N,N}$. By expanding the numerator of $\bar{\Psi}(x,y)$ in exponential form and using again the Paley–Wiener Theorem, we have

$$\begin{aligned}\psi_{N,N} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\bar{\Psi}\left(x,y\right)e^{-\mathrm{i}N(x+y)}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{16\pi^2\beta}\underbrace{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{-\mathrm{i}N(x+y)}}{2\kappa-\cos(x)-\cos(y)}\,\mathrm{d}x\,\mathrm{d}y}_{I(N)} + R,\end{aligned} \quad (34)$$

where, for all $\varepsilon > 0$, and R is a real number satisfying $|R| \leq L(\varepsilon)\left(\kappa-\sqrt{\kappa^2-1}\right)^{N(3-\varepsilon)}$ with $L(\varepsilon)$ being a positive real constant depending only on ε. Finally, by virtue of Lemma 7, the integral $I(N)$ features the large N asymptotic estimate

$$I(N) \sim \frac{\xi}{\sqrt{N}}\left(\kappa-\sqrt{\kappa^2-1}\right)^{2N}, \quad (35)$$

where $\xi := 1/(2\sqrt{\pi\kappa}(\kappa^2-1)^{1/4})$. Thus, from the latter estimate and the bounds in (33) and (34), it follows that, for large N, (31) has the asymptotics

$$\begin{aligned}\mathcal{W} &\sim 2\alpha^{2(N-2)}(4\psi_{N,N})\\ &\sim 8\alpha^{2(N-2)}I(N)\\ &\sim \frac{\xi\alpha^{2(N-2)}}{2\pi^2\beta\sqrt{N}}\left(\kappa-\sqrt{\kappa^2-1}\right)^{2N}.\end{aligned} \quad (36)$$

After some rearranging, the above expression yields the large n asymptotics (25). ■

As a consequence of Theorem 6, we have the following immediate result that characterizes the values of the parameters α, β, γ for which $\mathcal{W}$ either converges to zero or grows unbounded as the network dimension n increases.

Corollary 8: ***(Asymptotic transition for*** $\mathcal{K} = \{1\}$ ***and*** $\mathcal{T} = \{n\}$***)*** Consider the output controllability Gramian (3) where A is as in (4), and B and C are as in (2). If $\mathcal{K} = \{1\}$ and $\mathcal{T} = \{n\}$, then

$$\lim_{n\to\infty}\mathcal{W} = \begin{cases}\infty & \text{if } \omega(A) > 0,\\ 0 & \text{if } \omega(A) \leq 0\end{cases} \quad (37)$$

where $\omega(A) = \lambda_{\max}((A + A^\top)/2)$.

Proof: From Theorem 6, we have

$$\lim_{n\to\infty}\mathcal{W} = \begin{cases}\infty & \text{if } \alpha > \kappa+\sqrt{\kappa^2-1},\\ 0 & \text{if } 0 < \alpha \leq \kappa+\sqrt{\kappa^2-1}.\end{cases} \quad (38)$$

where we used the identity $\left(\kappa+\sqrt{\kappa^2-1}\right)^{-1} = \left(\kappa-\sqrt{\kappa^2-1}\right)$. As $n \to \infty$, it holds

$$\begin{aligned}\omega(A) &= \gamma + \beta\alpha + \beta/\alpha\\ &= \frac{\beta}{\alpha}\left(\alpha^2 - 2\kappa\alpha + 1\right).\end{aligned} \quad (39)$$

Thus, if $\omega(A) > 0$ then $\alpha^2 - 2\kappa\alpha + 1 > 0$ which in turn yields $\alpha > \kappa + \sqrt{\kappa^2-1}$. Conversely, if $\omega(A) \leq 0$ then $\alpha^2 - 2\kappa\alpha + 1 \leq 0$ which in turn yields $\alpha \leq \kappa + \sqrt{\kappa^2-1}$. Equation (37) now follows from the latter observations and (38). ■

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