Sandwich Bounds for Gramian and Hankel Spectra

Giacomo Baggio[1]

***Abstract*— We derive upper and lower bounds for the eigenvalues of the controllability and observability Gramians, as well as for the Hankel singular values, of continuous-time single-input single-output LTI systems with diagonalizable state matrix. The analysis employs a Cauchy matrix formulation that yields explicit, easily computable bounds in terms of the state matrix eigenvalues. The results provide insight into how the Gramian and Hankel spectra decay with the system size. Illustrative examples on structured systems show how the derived bounds are able to capture distinct spectral-scaling regimes.**

I. INTRODUCTION

The controllability and observability Gramians are cornerstone tools for analyzing linear time-invariant (LTI) systems. Their eigenvalues describe how control and output energy are distributed across the state space: directions associated with large eigenvalues are easily excitable or observable, while those associated with small eigenvalues are weakly controllable or observable. Consequently, the Gramian eigenvalues provide compact, physically interpretable summaries of the "energetic" structure of LTI systems [13], [16], [21]. These spectral features underpin a broad range of applications, including control-energy analysis [18], [23], [3], state-estimation performance [22], information-transmission efficiency [4], and the learnability of LTI models [20].

Despite their central role, analytical understanding of the dependence of Gramian eigenvalues on system dimension and structural features is still limited. Most existing results address upper bounds for individual eigenvalues, particularly the smallest eigenvalue of the controllability Gramian [18], [17], [3], [25], while only a few works provide more refined characterizations through lower bounds or asymptotic expressions [24], [14], [5]. Broader analyses covering the full Gramian spectrum have also been developed [19], [2], [6], [7], but these results remain mostly confined to upper-bound descriptions, with (informative) lower bounds still lacking.

Beyond their role in energy-based controllability and observability analysis, characterizing the full spectrum of the Gramians is also fundamental for model reduction. The eigenvalues of the product of the controllability and observability Gramians define the (squared) Hankel singular values, which quantify how strongly each state direction contributes to the input-output behavior of the system. The Hankel singular values govern the approximation error of optimal model reduction techniques, thereby setting fundamental limits on the achievable accuracy of reduced-order models [11], [1].

This paper develops upper and lower bounds for the eigenvalues of the controllability and observability Gramians, as well as for the Hankel singular values, of continuous-time single-input single-output LTI systems with diagonalizable state matrix. The analysis is based on a Cauchy-matrix representation of the Gramians, which leads to explicit and easily computable bounds expressed directly in terms of the state matrix spectrum. These expressions elucidate how the Gramian and Hankel spectra scale with the system dimension, and their behavior is illustrated on structured examples that reveal distinct scaling regimes, ranging from exponential to polynomial decay as the system size increases.

While the derived upper bounds share some similarities with those reported in [2] and related studies, they are established via different analytical techniques. More importantly, the proposed lower bounds constitute, to the best of the author's knowledge, the first explicit results of this nature, offering new insight and fundamental limits on the scaling of Gramian eigenvalues and Hankel singular values.

Notation. For a matrix $A \in \mathbb{C}^{n\times m}$, $A^\top$ denotes its transpose, A^* its conjugate transpose, and A_{ij} its (i, j) entry. If $A \in \mathbb{R}^{n\times n}$ has real eigenvalues, we denote its i-th eigenvalue by $\lambda_i(A)$ and, unless otherwise specified, assume they are ordered nonincreasingly, i.e., $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. We also write $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ for the largest and smallest eigenvalues of A, respectively. Given index sets P, Q, the submatrix of A with rows in P and columns in Q is written $A[P, Q]$; when $P = Q$, we simply write $A[P]$ for the corresponding principal submatrix. The trace of A is denoted $\mathrm{tr}(A)$ and the spectral norm of A by $\|A\|$. We write $A \sim B$ to indicate that two square matrices A, B are similar. For a (row or column) n-dimensional vector a, $\mathrm{diag}(a)$ denotes the diagonal matrix with diagonal entries given by a. $\mathrm{tridiag}(a, b, c) \in \mathbb{R}^{n\times n}$ denotes the tridiagonal matrix with sub-, main-, super-diagonal entries a, b, c, respectively. For symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, $A \succeq B$ denotes that $A - B$ is positive semidefinite. The symbol $e_i \in \mathbb{R}^n$ denotes the i-th canonical vector of $\mathbb{R}^n$. Finally, for a set S, $|S|$ denotes the cardinality of S and $\mathrm{Re}(x)$ the real part of $x \in \mathbb{C}$.

II. PRELIMINARIES

In this work, we consider continuous-time, LTI single-input single-output (SISO) systems of the form:

$$\begin{cases} \dot{x}(t) = A\,x(t) + b\,u(t), \\ y(t) = c\,x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$ denote the state, input, and output of the system, respectively. The matrices

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[1] G. Baggio is with the Department of Information Engineering, University of Padova, Padova 35131 Italy. E-mail: `baggio@dei.unipd.it`. This work was partially supported by the Italian Ministry of University and Research under the PRIN project "Extracting essential information and dynamics from complex networks", grant no. 2022MBC2EZ.

$A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, and $c \in \mathbb{R}^{1\times n}$ are the system, input, and output matrices, respectively. The input-output behavior of the system is compactly described by its transfer function $G(s) = c(sI - A)^{-1}b$.

Throughout, we make the following assumption.[1]

Assumption 1: The state matrix A is Hurwitz stable, i.e., all its eigenvalues have strictly negative real part, and diagonalizable. In addition, the pair (A, b) is controllable and the pair (A, c) is observable.

The (infinite-horizon) *controllability Gramian* associated with system (1) is defined as

$$\mathcal{W} = \int_0^\infty e^{At} bb^\top e^{A^\top t}\, \mathrm{d}t, \tag{2}$$

and it is positive definite if and only if the system is controllable. The controllability Gramian $\mathcal{W}$, and in particular its inverse, quantifies the input energy required to drive the system state from the origin to a prescribed configuration [3].

Dually, the (infinite-horizon) *observability Gramian* is

$$\mathcal{O} = \int_0^\infty e^{A^\top t} c^\top c\, e^{At}\, \mathrm{d}t, \tag{3}$$

which is positive definite if and only if the system is observable. The observability Gramian quantifies how the initial state influences the output energy of the system in the absence of external inputs, see e.g. [8].

Scalar quantities derived from the eigenvalues of $\mathcal{W}$ and $\mathcal{O}$ provide compact measures of the dynamic properties of the system: the spectrum of $\mathcal{W}$ reflects how much input energy is required to move the state in different directions, whereas the spectrum of $\mathcal{O}$ indicates how strongly those directions influence the output signal.

The *Hankel operator* of an LTI system maps past inputs to future outputs. For the system (1), it is given by

$$(\mathcal{H}u_-)(t) = c\,e^{At} \int_{-\infty}^0 e^{-A\tau} b\, u_-(\tau)\, \mathrm{d}\tau, \quad t > 0,$$

where u_- denotes the past input ($t \le 0$). For systems that are controllable, observable, and stable, $\mathcal{H}$ is compact and its nonzero singular values, the *Hankel singular values*, quantify the joint controllability-observability energy of each state direction. Such singular values satisfy [1]

$$\sigma_i = \sqrt{\lambda_i(\mathcal{W}\mathcal{O})}, \quad i = 1, \ldots, n, \tag{4}$$

where $\mathcal{W}$ and $\mathcal{O}$ are the controllability and observability Gramians defined in (2)-(3).

The Hankel singular values play a central role in model reduction, particularly in *balanced truncation* [1]. In balanced coordinates, the controllability and observability Gramians coincide and are diagonal,

$$\mathcal{W} = \mathcal{O} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \tag{5}$$

so that each state contributes independently with energy level σ_i. States with small σ_i can be truncated, yielding a reduced model of order $r < n$, described by transfer function $\tilde{G}(s)$, that preserves the dominant input-output behavior. Approximation quality is measured by the standard system norms:

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(\mathrm{i}\omega)\|, \;\; \|G\|^2_{\mathcal{H}_2} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(\mathrm{i}\omega)\|^2_{\mathrm{F}}\, \mathrm{d}\omega,$$

where $\|\cdot\|$ and $\|\cdot\|_{\mathrm{F}}$ denote the spectral and Frobenius norm, respectively. Classical balanced truncation error bounds are [1]

$$\|G - \tilde{G}\|_{\mathcal{H}_\infty} \le 2\sum_{i=r+1}^{n} \sigma_i, \quad \|G - \tilde{G}\|^2_{\mathcal{H}_2} \le 2\sum_{i=r+1}^{n} \sigma_i^2.$$

Thus, the Hankel singular values directly determine the achievable reduction accuracy.

The connection between Hankel singular values and approximation limits is even more direct in the case of *optimal Hankel-norm approximation*. The *Hankel norm* of a stable system is

$$\|G\|_{\text{Hankel}} := \sqrt{\lambda_{\max}(\mathcal{W}\mathcal{O})} = \sigma_1,$$

i.e., the largest Hankel singular value. A classic result states [11]

$$\min_{\tilde{G},\, \text{order}(\tilde{G})=r} \|G - \tilde{G}\|_{\text{Hankel}} = \sigma_{r+1}. \tag{6}$$

Hence, in this context, the Hankel singular values exactly characterize the attainable reduction error.

III. SANDWICH BOUNDS ON GRAMIAN EIGENVALUES

In the following, we focus on the controllability Gramian $\mathcal{W}$; however, all results apply to the observability Gramian $\mathcal{O}$ after suitable adaptations.

We begin with a result that establishes an explicit relationship between the controllability Gramian and a Cauchy matrix determined by the eigenvalues of A.

Lemma 1: ***(Gramian and Cauchy matrix)*** Consider the system (1). Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of A and V the eigenvector matrix of A. Define $Q := V\mathrm{diag}(V^{-1}b)$. Then:

(i) The controllability Gramian $\mathcal{W}$ admits the factorization

$$\mathcal{W} = Q\mathcal{C}Q^*, \tag{7}$$

where $\mathcal{C} \in \mathbb{C}^{n\times n}$ is the Cauchy matrix with entries

$$\mathcal{C}_{ij} = -\frac{1}{\lambda_i + \lambda_j^*}.$$

(ii) For all $k = 1, \ldots, n$,

$$\frac{1}{\|Q^{-1}\|^2}\lambda_k(\mathcal{C}) \;\le\; \lambda_k(\mathcal{W}) \;\le\; \|Q\|^2\lambda_k(\mathcal{C}). \tag{8}$$

Proof: (i) The decomposition (7) is a classic result and follows from the diagonalization of A via V and the explicit integral expression of the Gramian; see, e.g., [1, Lemma 9.4].

(ii) For any nonzero $x \in \mathbb{C}^n$, set $y := Q^*x$. Then

$$x^*\mathcal{W}x = x^*Q\mathcal{C}Q^*x = y^*\mathcal{C}y.$$

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[1] While Hurwitz stability, controllability, and observability are standard system-theoretic assumptions, the diagonalizability of A is here introduced to simplify the analysis of Gramians.

Moreover,

$$\frac{1}{\|Q^{-1}\|^2} x^* x \;\leq\; x^* Q Q^* x \;=\; y^* y \;\leq\; \|Q\|^2 x^* x,$$

which implies

$$\frac{1}{\|Q^{-1}\|^2} \frac{y^* \mathcal{C} y}{y^* y} \;\leq\; \frac{x^* \mathcal{W} x}{x^* x} \;\leq\; \|Q\|^2 \frac{y^* \mathcal{C} y}{y^* y}.$$

By the Courant-Fischer min-max characterization of eigenvalues [12, Theorem 4.2.6], this yields (8). ■

Building on Lemma 1, we next derive explicit spectral bounds for the controllability Gramian in terms of the eigenvalues of A.

Theorem 1: ***(Bounds on Gramian eigenvalues)*** Consider the system (1), and let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of A. For any nonempty subset $S \subseteq \{1, \ldots, n\}$ and $i \in S$, define

$$\theta_i(S) \;:=\; \frac{1}{2|\mathrm{Re}(\lambda_i)|} \prod_{j \in S \setminus \{i\}} \left| \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j^*} \right|^2. \tag{9}$$

Then, for each $k = 1, \ldots, n$, it holds

$$\lambda_k(\mathcal{W}) \geq c_{\min} \max_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \min_{i \in S} \theta_i(S), \tag{10}$$

$$\lambda_k(\mathcal{W}) \leq c_{\max} \min_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \max_{i \in T} \theta_i\big(\overline{T} \cup \{i\}\big), \tag{11}$$

where $c_{\min} := 1/(\|Q^{-1}\|^2 k)$ and $c_{\max} := \|Q\|^2 (n-k+1)$ with Q defined in Lemma 1, and $\overline{T}$ is the complement of T in $\{1, \ldots, n\}$.

Proof: We start with the lower bound. By Lemma 1(ii),

$$\lambda_k(\mathcal{W}) \geq \frac{1}{\|Q^{-1}\|^2} \lambda_k(\mathcal{C}).$$

Applying Cauchy's interlacing theorem [12, Theorem 4.3.17] gives

$$\lambda_k(\mathcal{C}) \geq \max_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \lambda_k(\mathcal{C}[S]),$$

so that

$$\lambda_k(\mathcal{W}) \geq \frac{1}{\|Q^{-1}\|^2} \max_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \lambda_k(\mathcal{C}[S]).$$

For any positive definite matrix M, $\lambda_{\min}(M)^{-1} \leq \mathrm{tr}(M^{-1})$, hence

$$\lambda_k(\mathcal{W}) \geq \frac{1}{\|Q^{-1}\|^2} \max_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \frac{1}{\mathrm{tr}(\mathcal{C}[S]^{-1})}.$$

Using $\mathrm{tr}(M^{-1}) \leq k \max_i (M^{-1})_{ii}$ for a $k \times k$ invertible matrix M,

$$\lambda_k(\mathcal{W}) \geq \frac{1}{\|Q^{-1}\|^2 k} \max_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} \frac{1}{\max_{i \in S} (\mathcal{C}[S]^{-1})_{ii}}.$$

Finally, by using Lemma 2(ii) in Appendix A with $x_i = -\lambda_i$, $y_j = -\lambda_j^*$, we obtain an explicit expression for $(\mathcal{C}[S]^{-1})_{ii}$ which yields (10).

For the upper bound, consider $\mathcal{C}^{-1}$. Applying again Cauchy's interlacing theorem,

$$\lambda_{n-k+1}(\mathcal{C}^{-1}) \geq \max_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \lambda_{n-k+1}\big((\mathcal{C}^{-1})[T]\big).$$

Using again the fact that for any positive definite matrix M, $\lambda_{\min}(M)^{-1} \leq \mathrm{tr}(M^{-1})$,

$$\begin{aligned} \lambda_{n-k+1}(\mathcal{C}^{-1}) &\geq \max_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \frac{1}{\mathrm{tr}\big(((\mathcal{C}^{-1})[T])^{-1}\big)} \\ &= \frac{1}{\displaystyle\min_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \mathrm{tr}\big(((\mathcal{C}^{-1})[T])^{-1}\big)}. \end{aligned}$$

Since $\lambda_k(\mathcal{C}) = 1/\lambda_{n-k+1}(\mathcal{C}^{-1})$, we obtain

$$\lambda_k(\mathcal{W}) \leq \|Q\|^2 \lambda_k(\mathcal{C}) \leq \|Q\|^2 \min_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \mathrm{tr}\big(((\mathcal{C}^{-1})[T])^{-1}\big),$$

where the first inequality follows from Lemma 1(ii). Applying the trace bound again,

$$\lambda_k(\mathcal{W}) \leq \|Q\|^2 (n-k+1) \min_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \max_{i \in T} \big(((\mathcal{C}^{-1})[T])^{-1}\big)_{ii}.$$

Finally, substituting in the above inequality the explicit formula of Lemma 2(iii) in Appendix A with $x_i = -\lambda_i$, $y_j = -\lambda_j^*$ yields (11). ■

It is worth noting that the lower and upper bounds in Theorem 1 share a common structural form: both involve products of exactly $k-1$ eigenvalue-dependent factors, appearing respectively in the terms $\theta_i(S)$ and $\theta_i(\overline{T} \cup \{i\})$. This suggests that the bounds become comparably tight when the eigenvalue gaps of A are approximately uniform, so that all spectral factors contribute at a similar scale, and when the eigenvector matrix V of A is well-conditioned. In fact, the prefactors $c_{\min}$ and $c_{\max}$ depend on $\|Q^{-1}\|$ and $\|Q\|$, and thus reflect the conditioning of V; when this is moderate, they do not introduce significant imbalance in the bounds.

As a direct consequence of Theorem 1, we next derive explicit bounds for the largest and smallest eigenvalues of the controllability Gramian.

Corollary 1: ***(Bounds on extremal Gramian eigenvalues)*** Consider the system (1), and let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of A. Then:

(i) the maximum eigenvalue of $\mathcal{W}$ satisfies

$$\frac{1/\|Q^{-1}\|^2}{2 \min_i |\mathrm{Re}(\lambda_i)|} \leq \lambda_1(\mathcal{W}) \leq \frac{n \|Q\|^2}{2 \min_i |\mathrm{Re}(\lambda_i)|}, \tag{12}$$

where Q is defined in Lemma 1.

(ii) the minimum eigenvalue of $\mathcal{W}$ satisfies

$$\frac{1}{\|Q^{-1}\|^2 n} \theta_{\min} \leq \lambda_n(\mathcal{W}) \leq \|Q\|^2 \theta_{\min}, \tag{13}$$

where $\theta_{\min} := \min_{i=1,\ldots,n} \theta_i(\{1, \ldots, n\})$ and $\theta_i(\cdot)$ is defined in (9).

Proof: The result follows directly from Theorem 1 by substituting $k = 1$ for the upper and lower bounds on $\lambda_1(\mathcal{W})$, and $k = n$ for those on $\lambda_n(\mathcal{W})$. ■

Remark 1: ***(Computational considerations)*** The bounds in Theorem 1 require an optimization over all index subsets of a given cardinality, which generally entails combinatorial complexity. However, since each inequality in the theorem holds for every admissible subset S or T, the bounds remain valid even when these subsets are fixed rather than optimized. In this case, their evaluation becomes straightforward, as it involves computing (or optimizing over) at most n terms that depend only on the eigenvalues of A. This reformulation greatly simplifies the numerical evaluation of the bounds and makes their dependence on the eigenvalues of A through the factors θ_i more transparent. ▲

Beyond offering computationally tractable estimates for the Gramian eigenvalues, the bounds in Theorem 1 also provide analytical insight into how these eigenvalues decay as the system dimension increases, as we illustrate next.

Example 1: ***(Scaling of Gramian eigenvalues for a tridiagonal chain)*** Let

$$A = \text{tridiag}(1, -2, 1) \in \mathbb{R}^{n\times n}, \quad b = e_1.$$

The eigenpairs of A are given by (see e.g. [15, Ex. 7.2.5]):

$$\lambda_i = -4\sin^2\Big(\frac{i\pi}{2(n+1)}\Big),\ V_{ij} = \sqrt{\frac{2}{n+1}}\sin\Big(\frac{ij\pi}{n+1}\Big). \tag{14}$$

Note that V is orthogonal, so that $\|Q\|^2 = \max_i d_i^2$, $1/\|Q^{-1}\|^2 = \min_i d_i^2$, with $d_i := (V^\top b)_i$. Moreover, from (14),

$$(V^\top b)_i^2 = \frac{2}{n+1}\sin^2\Big(\frac{i\pi}{n+1}\Big).$$

From $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \pi/2]$ and $\sin x \leq 1$ and the definition of c_{min}, c_{max}, it holds[2]

$$c_{\text{min}} \geq \frac{8}{k(n+1)^3} \geq \frac{8}{n(n+1)^3}, \tag{15}$$

$$c_{\text{max}} \leq \frac{2(n-k+1)}{n+1} \leq \frac{2n}{n+1}. \tag{16}$$

Upper bound. Assume $k \geq 2$. From (11), take $T = \{k, \dots, n\}$. The behavior of $\theta_i(\overline{T} \cup \{i\})$ for this choice is analyzed in Appendix B-1), where it is shown that, for all $i \in T$,

$$\theta_i(\overline{T} \cup \{i\}) \leq \frac{(n+1)^2}{8k^2} e^{-\rho(k-1)},$$

where $\rho := \frac{3}{8}\frac{k^2}{n^2}$. Therefore, from (11) and (16),

$$\begin{aligned}\lambda_k(\mathcal{W}) &\leq c_{\text{max}} \max_{i\in T} \theta_i(\overline{T} \cup \{i\}) \\ &\leq C_{\text{max}}(n)\, e^{-\rho k},\end{aligned}$$

for some positive constant $C_{\text{max}}(n)$ that depends polynomially on n.

Lower bound. Assume $k \geq 2$. Using (10) with $S = \{1, \dots, k\}$, in Appendix B-2) the following lower bound on each $\theta_i(S)$, $i \in S$, is derived

$$\theta_i(S) \geq \frac{1}{8} e^{-\eta(k-1)},$$

where $\eta := 2\log(4\pi e)$. Hence, from (10) and (15),

$$\lambda_k(\mathcal{W}) \geq C_{\text{min}}(n)\, e^{-\eta k},$$

for some positive constant $C_{\text{min}}(n)$ that depends polynomially on n.

Asymptotic scaling. When k grows linearly with n, i.e. $k = cn$, $c \in (0, 1]$, it holds $\rho = \frac{3}{8}c^2$ is a positive constant independent of n.[3] In this regime, the bounds above imply that

$$C_{\text{min}}(n)e^{-\eta cn} \ \leq\ \lambda_k(\mathcal{W}) \ \leq\ C_{\text{max}}(n)e^{-\rho cn},$$

that is, $\lambda_k(\mathcal{W})$ decays *exponentially* with n.

Observe also that when k grows sublinearly with n, the lower bound implies that $\lambda_k(\mathcal{W})$ cannot decay exponentially in n, and thus the decay is at most subexponential as $n \to \infty$. In particular, when $k = \log n$, the lower bound yields

$$\lambda_k(\mathcal{W}) \geq C_{\text{min}}(n)e^{-\eta\log n} = C_{\text{min}}(n)\, n^{-\eta},$$

so that the decay is at most polynomial in n. ◇

IV. SANDWICH BOUNDS ON HANKEL SINGULAR VALUES

As recalled in Section II, the Hankel singular values of system (1) are the square roots of the eigenvalues of the product of the controllability and observability Gramians. Because they depend directly on the spectrum of this product, one expects a close correspondence between the bounds derived above and those governing the Hankel singular values. This correspondence is formalized in the following result.

Theorem 2: ***(Bounds on Hankel singular values)*** Consider the system (1) and let $\{\sigma_k\}_{k=1}^n$ denote its Hankel singular values ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Moreover, let V be the eigenvector matrix of A. Define $d := V^{-1}b$, $f := cV$, and $r_i := d_i f_i$ for $i = 1, \dots, n$. If all r_i are real and share the same sign, then, for each $k = 1, \dots, n$, it holds

$$\sigma_k \geq q_{\text{min}} \max_{\substack{S\subseteq\{1,\dots,n\} \\ |S|=k}} \min_{i\in S} \theta_i(S), \tag{17}$$

$$\sigma_k \leq q_{\text{max}} \min_{\substack{T\subseteq\{1,\dots,n\} \\ |T|=n-k+1}} \max_{i\in T} \theta_i\big(\overline{T} \cup \{i\}\big), \tag{18}$$

where $q_{\text{min}} := r_{\text{min}}/k$, $q_{\text{max}} := r_{\text{max}}(n-k+1)$ with $r_{\text{min}} := \min_i |r_i|$, $r_{\text{max}} := \max_i |r_i|$, and $\theta_i(\cdot)$ is defined in Theorem 1.

Proof: By Lemma 1(i), the controllability Gramian admits the factorization

$$\mathcal{W} = Q\mathcal{C}Q^*,$$

where $Q = V\text{diag}(V^{-1}b)$ and $\mathcal{C}$ is the Cauchy matrix with entries $\mathcal{C}_{ij} = -1/(\lambda_i + \lambda_j^*)$. By adapting Lemma 1(i) we obtain a similar factorization for the observability Gramian,

$$\mathcal{O} = L^*\mathcal{C}L,$$

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[2] For $i = 1$, $\sin(\frac{\pi}{n+1}) \geq \frac{2}{n+1}$, giving $\min_i d_i^2 \geq \frac{8}{(n+1)^3}$, while $\max_i d_i^2 \leq \frac{2}{n+1}$ readily follows from $\sin x \leq 1$.

[3] One also checks that $\rho \leq \eta$ for all $c \in (0, 1]$. Indeed, $\rho \leq 3/8 \leq 2\log(4\pi e) = \eta$.

where $L = \mathrm{diag}(cV)V^{-1}$. Hence,

$$\mathcal{WO} = V\mathrm{diag}(V^{-1}b)\,\mathcal{C}\,R\,\mathcal{C}\,\mathrm{diag}(cV)V^{-1},$$

where $R = \mathrm{diag}(r_1, \dots, r_n)$.

Since $\mathrm{diag}(V^{-1}b)$, $\mathrm{diag}(cV)$ are invertible[4], and $\mathcal{C}$ is positive definite, it follows that $\mathcal{WO}$ is similar to

$$\mathcal{C}^{1/2}R\,\mathcal{C}\,R\,\mathcal{C}^{1/2},$$

where $\mathcal{C}^{1/2}$ denotes the principal (positive definite) square root of $\mathcal{C}$. Next, since, by assumption, all r_i are real and share the same sign, there exists $\varepsilon \in \{\pm 1\}$ such that $R = \varepsilon |R|$, where $|R| = \mathrm{diag}(|r_1|, \dots, |r_n|)$. Therefore,

$$\mathcal{C}^{1/2}R\,\mathcal{C}\,R\,\mathcal{C}^{1/2} = \mathcal{C}^{1/2}|R|\,\mathcal{C}\,|R|\,\mathcal{C}^{1/2} = (\mathcal{C}^{1/2}\,|R|\,\mathcal{C}^{1/2})^2.$$

Consequently, for all $k = 1, \dots, n$,

$$\sigma_k = \lambda_k(\mathcal{C}^{1/2}\,|R|\,\mathcal{C}^{1/2}).$$

Because $r_{\min} I \preceq |R| \preceq r_{\max} I$, it follows that

$$r_{\min}\,\lambda_k(\mathcal{C}) \le \sigma_k \le r_{\max}\lambda_k(\mathcal{C}).$$

The desired inequalities (17)-(18) then follow directly from the bounds on $\lambda_k(\mathcal{C})$ derived in the proof of Theorem 1. ■

The previous result shows that bounds similar to those obtained for Gramian eigenvalues also hold for the Hankel singular values, provided an additional condition on the scalar coefficients r_i is satisfied. The meaning of this assumption is clarified below.

Remark 2: The quantities $r_i = d_i f_i$ in Theorem 2 are precisely the *residues* of the transfer function $G(s) = c(sI - A)^{-1}b$. Indeed, using the eigendecomposition $A = V\Lambda V^{-1}$, with $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, we have

$$G(s) = cV(sI - \Lambda)^{-1}V^{-1}b = \sum_{i=1}^{n} \frac{r_i}{s - \lambda_i},$$

so that each r_i is the residue associated with the simple pole λ_i of $G(s)$. When all r_i are real and positive, the system is a *relaxation system*, that is, a strictly stable linear system whose transfer function is real, with real poles and positive residues. Such systems represent a broad class of dissipative or diffusive models, for example RC networks, thermal systems, and viscoelastic dynamics, whose impulse responses are nonnegative sums of decaying exponentials [10]. ▲

We next revisit the tridiagonal system of Example 1 and apply Theorem 2 to characterize the scaling of its Hankel singular values.

Example 2: ***(Scaling of Hankel singular values for a tridiagonal chain)*** Let

$$A = \mathrm{tridiag}(1, -2, 1) \in \mathbb{R}^{n\times n},\ b = e_1,\ c = e_1^\top + \tfrac{1}{2}e_n^\top.$$

From the explicit eigenpair expressions of A in (14), it follows that

$$d_i = (V^\top e_1)_i = \sqrt{\frac{2}{n+1}}\sin\Big(\frac{i\pi}{n+1}\Big)$$
$$(e_n^\top V)_i = (-1)^{i+1}d_i,$$

so that

$$f_i = ((e_1^\top + \tfrac{1}{2}e_n^\top)V)_i = d_i\Big(1 - \tfrac{1}{2}(-1)^i\Big).$$

Therefore, with $\beta_i := 1 - \frac{1}{2}(-1)^i \in \{\frac{1}{2}, \frac{3}{2}\}$ and $m_i := \min\{i, n+1-i\}$,

$$r_i = d_i f_i = \frac{2}{n+1}\sin^2\Big(\frac{\pi m_i}{n+1}\Big)\,\beta_i > 0.$$

From $\sin x \ge \frac{2}{\pi}x$ and $\sin x \le x$ on $[0, \pi/2]$ and the definition of $q_{\min}$, $q_{\max}$, we obtain

$$q_{\min} \ge \frac{4}{k(n+1)^3} \ge \frac{4}{n(n+1)^3},$$
$$q_{\max} \le \frac{3(n-k+1)}{n+1} \le \frac{3n}{n+1}.$$

By applying Theorem 2, we obtain results analogous to those in Example 1. In particular, when $k = cn$ with $c \in (0, 1]$, the Hankel singular values σ_k decay exponentially with n. Conversely, when k grows sublinearly with n, the lower bound ensures a subexponential decay, and in the specific case $k = \log n$, the decay is at most polynomial in n. Since the Hankel singular values determine the optimal Hankel-norm approximation error through (6), these results also describe how the achievable model reduction error scales with the system dimension. ◇

V. CONCLUSIONS

This work has derived explicit upper and lower bounds for the eigenvalues of the controllability and observability Gramians, as well as for the Hankel singular values of SISO LTI systems with diagonalizable state matrix. The proposed Cauchy-matrix formulation provides closed-form expressions that make the dependence of these spectral quantities on the state matrix eigenvalues explicit. The resulting bounds offer both computationally efficient estimates and analytical insight into the scaling laws that govern how the Gramian and Hankel spectra, and thus the achievable accuracy of optimal model reduction techniques, vary with the system dimension.

Future work will focus on extending these results to multi-input multi-output systems, on relaxing the assumption on residues of Theorem 2, and on deriving exact asymptotic decay rates, as recently obtained in [5] for the smallest eigenvalue of the controllability Gramian.

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APPENDIX

A. Auxiliary results on Cauchy matrices

We recall that, for a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $A[P, Q]$ the submatrix obtained by selecting the rows of A indexed by P and the columns indexed by Q. When $P = Q$, we simply write $A[P]$; in this case, $A[P]$ is the principal submatrix of A indexed by P.

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[4] This follows from the standing controllability/observability assumptions.

Lemma 2: ***(Principal submatrices of a Cauchy matrix)*** Let $\mathcal{C} \in \mathbb{C}^{n\times n}$ be a Cauchy matrix with entries

$$\mathcal{C}_{ij} = \frac{1}{x_i + y_j}, \quad x_i + y_j \neq 0, x_i \neq x_j, y_i \neq y_j \text{ for } i \neq j.$$

Let $S \subseteq \{1,\dots,n\}$ with $|S| = k$, and let $T = \overline{S} = \{1,\dots,n\} \setminus S$. Without loss of generality, assume $S = \{1,\dots,k\}$ and $T = \{k+1,\dots,n\}$. Then:

(i) $\mathcal{C}[S]$ is a Cauchy matrix with

$$(\mathcal{C}[S])_{ij} = \frac{1}{x_i + y_j}, \quad i,j = 1,\dots,k.$$

(ii) $\mathcal{C}[S]$ is invertible and

$$(\mathcal{C}[S]^{-1})_{ij} = (x_j + y_i) \prod_{\ell=1,\ell\neq i}^{k} \frac{x_j + y_\ell}{y_i - y_\ell} \prod_{\ell=1,\ell\neq j}^{k} \frac{x_\ell + y_i}{x_j - x_\ell}.$$

(iii) $(\mathcal{C}^{-1})[S]$ is invertible and

$$\big((\mathcal{C}^{-1})[S]\big)^{-1} = D_x\, \mathcal{C}[S]\, D_y,$$

where $D_x = \operatorname{diag}(\alpha_1,\dots,\alpha_k)$, $\alpha_i = \prod_{\ell=k+1}^{n} \frac{x_\ell - x_i}{x_\ell + y_i}$, and $D_y = \operatorname{diag}(\beta_1,\dots,\beta_k)$, $\beta_j = \prod_{\ell=k+1}^{n} \frac{y_\ell - y_j}{x_j + y_\ell}$.

Proof: (i) Restricting $\mathcal{C}$ to $S = \{1,\dots,k\}$ gives

$$(\mathcal{C}[S])_{ij} = \frac{1}{x_i + y_j}, \quad i,j = 1,\dots,k,$$

so $\mathcal{C}[S]$ is again a Cauchy matrix with parameters $x_1,\dots,x_k$ and $y_1,\dots,y_k$.

(ii) It is well known that Cauchy matrices with distinct parameters are nonsingular and admit an explicit inverse formula (see, e.g., [9, Fact 3.20.14]). Applying this formula to $\mathcal{C}[S]$ gives the stated expression.

(iii) Since $\mathcal{C}$ and all its principal submatrices are Cauchy matrices with distinct parameters, they are all invertible. Hence, $(\mathcal{C}^{-1})[S]$ is invertible because

$$\det\big((\mathcal{C}^{-1})[S]\big) = \frac{\det(\mathcal{C}[T])}{\det(\mathcal{C})} \neq 0,$$

by the Schur determinant formula [12, Sec. 0.8.5].

We now compute $\big((\mathcal{C}^{-1})[S]\big)^{-1}$. For $i,j \in S$, the inverse formula applied to $\mathcal{C}$ gives

$$(\mathcal{C}^{-1})_{ij} = (x_j + y_i) \prod_{\ell=1,\ell\neq i}^{n} \frac{x_j + y_\ell}{y_i - y_\ell} \prod_{\ell=1,\ell\neq j}^{n} \frac{x_\ell + y_i}{x_j - x_\ell}.$$

Splitting the products over S and T, we obtain

$$(\mathcal{C}^{-1})_{ij} = (x_j + y_i) \prod_{\ell\in S,\ell\neq i} \frac{x_j + y_\ell}{y_i - y_\ell} \prod_{\ell\in S,\ell\neq j} \frac{x_\ell + y_i}{x_j - x_\ell} \cdot \prod_{\ell\in T} \frac{x_j + y_\ell}{y_i - y_\ell} \prod_{\ell\in T} \frac{x_\ell + y_i}{x_j - x_\ell}.$$

Note that

$$\prod_{\ell\in T} \frac{x_j + y_\ell}{y_i - y_\ell} \prod_{\ell\in T} \frac{x_\ell + y_i}{x_j - x_\ell} = \prod_{\ell\in T} \frac{x_\ell + y_i}{x_\ell - x_i} \prod_{\ell\in T} \frac{x_j + y_\ell}{y_\ell - y_j} = \frac{1}{\alpha_i \beta_j}.$$

Hence

$$(\mathcal{C}^{-1})_{ij} = \frac{1}{\alpha_i\beta_j}(x_j + y_i) \prod_{\ell\in S,\ell\neq i} \frac{x_j + y_\ell}{y_i - y_\ell} \prod_{\ell\in S,\ell\neq j} \frac{x_\ell + y_i}{x_j - x_\ell}.$$

By part (ii), the last factor equals $(\mathcal{C}[S]^{-1})_{ij}$, so

$$(\mathcal{C}^{-1})[S] = D_x^{-1}\, \mathcal{C}[S]^{-1}\, D_y^{-1}.$$

Since D_x and D_y are invertible, we conclude that $\big((\mathcal{C}^{-1})[S]\big)^{-1} = D_x\, \mathcal{C}[S]\, D_y$. ■

B. Bounds on $\theta_i(S)$ for the tridiagonal chain

This section provides the bounds for $\theta_i(S)$ used in Example 1. Recall that

$$\theta_i(S) = \frac{1}{2|\lambda_i|} \prod_{j\in S\setminus\{i\}} \Big(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\Big)^2, \;\; \lambda_i = -4\sin^2\Big(\frac{i\pi}{2(n+1)}\Big),$$

and that $|\lambda_i|$ increases monotonically with i.

1) Upper bound on $\theta_i(\overline{T} \cup \{i\})$: Let $T = \{k,\dots,n\}$ and fix $i \in T$. Define $r_{ij} := \frac{|\lambda_j|}{|\lambda_i|} \in (0,1)$ for $j < k$. Using $\log\frac{1-r}{1+r} = -2\operatorname{atanh}(r)$, we have

$$\begin{aligned}\theta_i(\overline{T} \cup \{i\}) &= \frac{1}{2|\lambda_i|} \prod_{j<k} \Big(\frac{1 - r_{ij}}{1 + r_{ij}}\Big)^2 \\ &= \frac{1}{2|\lambda_i|} e^{-4\sum_{j<k} \operatorname{atanh}(r_{ij})}.\end{aligned}$$

Since $\operatorname{atanh}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} \cdots \geq x$ for $x \in [0,1)$,

$$\theta_i(\overline{T} \cup \{i\}) \leq \frac{1}{2|\lambda_i|} e^{-4\sum_{j<k} r_{ij}}. \tag{19}$$

We now lower bound the sum in the exponent. Let $\alpha := \frac{\pi}{2(n+1)}$. Then $|\lambda_\ell| = 4\sin^2(\ell\alpha)$, and note that $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \frac{\pi}{2}]$ and $\sin x \leq 1$, which imply

$$|\lambda_\ell| \geq 4\Big(\frac{2}{\pi}\,\ell\alpha\Big)^2 = \frac{4\ell^2}{(n+1)^2}, \quad |\lambda_\ell| \leq 4.$$

Hence, for every $j < k \leq i$,

$$r_{ij} = \frac{|\lambda_j|}{|\lambda_i|} \;\geq\; \frac{|\lambda_j|}{4} \;\geq\; \frac{j^2}{(n+1)^2}.$$

Summing over $j = 1,\dots,k-1$ and using $\sum_{j=1}^{k-1} j^2 = \frac{(k-1)k(2k-1)}{6} \geq \frac{(k-1)^3}{6}$,

$$\sum_{j<k} r_{ij} \;\geq\; \frac{1}{(n+1)^2}\frac{(k-1)^3}{6}. \tag{20}$$

Insert (20) in (19) to obtain

$$\theta_i(\overline{T} \cup \{i\}) \leq \frac{1}{2|\lambda_i|}\, e^{-\frac{2}{3}\frac{(k-1)^3}{(n+1)^2}}. \tag{21}$$

Finally, since $i \geq k$ and $|\lambda_i|$ increases with i,

$$\frac{1}{2|\lambda_i|} \leq \frac{1}{2|\lambda_k|}, \quad |\lambda_k| = 4\sin^2\Big(\frac{k\pi}{2(n+1)}\Big) \geq \frac{4k^2}{(n+1)^2},$$

whence

$$\frac{1}{2|\lambda_k|} \;\leq\; \frac{(n+1)^2}{8k^2}. \tag{22}$$

From (21) and (22), it follows that

$$\theta_i(\overline{T} \cup \{i\}) \le \frac{(n+1)^2}{8k^2} e^{-\frac{2}{3}\frac{(k-1)^3}{(n+1)^2}} \\ \le \frac{(n+1)^2}{8k^2} e^{-\frac{3}{8}\frac{k^2}{n^2}(k-1)}$$

where in the last step we used $k-1 \ge k/2$ for $k \ge 2$ and $n+1 \le 3n/2$ for $n \ge 2$.

2) Lower bound on $\theta_i(S)$: Let $S = \{1, \ldots, k\}$ and $\alpha := \frac{\pi}{2(n+1)}$. Using the bound[5]

$$|\sin^2(\tfrac{\pi}{2}x) - \sin^2(\tfrac{\pi}{2}y)| \ge (x-y)^2, \;\; x, y \in [0,1],$$

we have for all i, j,

$$|\sin^2(i\alpha) - \sin^2(j\alpha)| \ge \frac{1}{(n+1)^2}(i-j)^2.$$

From $\sin(x) \le x$ for $x \ge 0$, it follows that

$$\prod_{j=1, j\ne i}^{k} \Big(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\Big)^2 \ge \left(\frac{2}{\pi}\right)^{2(k-1)} \prod_{j=1, j\ne i}^{k} \left(\frac{i-j}{i+j}\right)^2,$$

Now $\prod_{j\ne i} |i-j| = (i-1)!(k-i)!$, while for each $j \ne i$ we have $i + j \le i + k \le 2k$, hence

$$\prod_{j\ne i}(i+j) \;\le\; (2k)^{k-1}.$$

Therefore,

$$\prod_{j=1, j\ne i}^{k} \Big(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\Big)^2 \;\ge\; \Big(\frac{(i-1)!(k-i)!}{(\pi k)^{k-1}}\Big)^2.$$

Using the inequality $(i-1)!(k-i)! \ge \frac{(k-1)!}{2^{k-1}}$,[6] we obtain

$$\prod_{j=1, j\ne i}^{k} \Big(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\Big)^2 \;\ge\; \frac{(k-1)!^2}{(2\pi k)^{2(k-1)}}.$$

Finally, since $|\lambda_i| \le 4$,

$$\theta_i(S) \;\ge\; \frac{1}{8}\,\frac{(k-1)!^2}{(2\pi k)^{2(k-1)}}.$$

Using Stirling's inequality $(k-1)! \ge \left(\frac{k-1}{e}\right)^{k-1}$ yields the expression

$$\theta_i(S) \ge \frac{1}{8}\left(\frac{k-1}{2\pi e\, k}\right)^{2(k-1)} \ge \frac{1}{8}\left(\frac{1}{4\pi e}\right)^{2(k-1)},$$

where in the last step we used $k-1 \ge k/2$ for $k \ge 2$.

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[5] For $x, y \in [0,1]$, let $a = \frac{\pi}{2}x$, $b = \frac{\pi}{2}y$, and $d = |a-b| = \frac{\pi}{2}|x-y|$. Since $a, b \in [0, \frac{\pi}{2}]$, we have $a + b \in [d, \pi - d]$, hence $|\sin^2 a - \sin^2 b| = |\sin(a+b)\sin(a-b)| \ge \sin^2 d = \sin^2(\frac{\pi}{2}|x-y|) \ge (x-y)^2$, using $\sin x \ge \frac{2}{\pi}x$ for $x \in [0, \frac{\pi}{2}]$.

[6] The inequality follows from $\binom{k-1}{i-1} = \frac{(k-1)!}{(i-1)!(k-i)!} \le 2^{k-1}$.

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